Sur certains processus aléatoires en milieu aléatoire

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Chapter 1

Introduction.

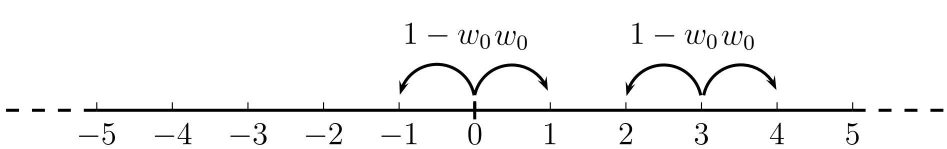
La motivation première pour étudier des processus aléatoires en milieu aléatoire est de répondre à la question naturelle : un processus dans un milieu inhomogène, mais régulier dans un certain sens a-t-il, au moins approximativement, les mêmes propriétés qu'un processus similaire dans un milieu homogène. Bien entendu cette formulation est assez imprécise, nous allons donc tout d'abord expliciter un peu notre propos.

Par inhomogénéité, nous entendons que la configuration locale de l'environnement sera aléatoire, et indépendante de la configuration dans des points éloignés de l'espace.

Cependant il est intéressant de supposer que cet environnement est une perturbation aléatoire d'un environnement régulier, pour traduire ceci nous ferons l'hypothèse qu'une certaine invariance par translation d'espace existe.

Le premier exemple d'étude de milieux aléatoires a été inspiré par la biophysique. En effet, en 1967, dans le but d'étudier la réplication de l'ADN, A.A. Chernov [17] introduisit un modèle très simple, appelé Marche aléatoire en milieu aléatoire sur $\mathbb{Z}$ (MAMA), qui peut être décrit de la façon suivante:

- on considère une famille de variables aléatoires i.i.d. $w_i, i \in \mathbb{N}$, telle que $w_i \in [0, 1]$.

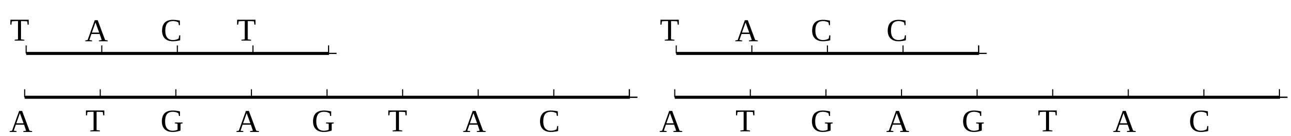


- Soit X_n la marche aléatoire définie par

$$\begin{cases} X_0 = 0, \\ P_w[X_{n+1} = x + 1 | X_n = x] = w_x, \\ P_w[X_{n+1} = x - 1 | X_n = x] = 1 - w_x. \end{cases}$$

Le lien entre ce processus et l'ADN n'est pas évident, donc nous allons expliquer comment cet "environnement" intervient. L'idée de Chernov est de modéliser la réplication de l'ADN de la façon suivante : la chaîne originale est constituée de molécules A,C,G,T, et une seconde chaîne est construite selon le principe suivant:

- Au départ la chaîne est vide, puis une molécule forme une paire avec la première molécule de la chaîne originale.
- Ensuite, à chaque étape, soit une molécule est ajoutée, soit la dernière est enlevée. Bien sûr, les "mauvaises paires" C-T ou A-G ont une plus grande chance d'être supprimées que les "bonnes" A-T et G-C. En effet, si à la fin de la réplication il reste une mauvaise paire dans la chaîne, alors une mutation apparaît.



faible chance de suppression grande chance de suppression

Les probabilités de formation et de suppression des paires sont toutes différentes, donc la longueur de la chaîne créée suit la loi d'une marche aléatoire en milieu aléatoire, le milieu étant donné par la chaîne originale.

Quelques années plus tard, D.E. Temkin [96], intéressé par des problèmes issus de la métallurgie, fait d'autres travaux sur le modèle.

De nombreux résultats mathématiques ont été obtenus depuis, en particulier par F. Solomon [89], Ya. G. Sinaï [88] et H. Kesten, M. V. Koslov et F. Spitzer [58].

Ce processus a été récemment l'objet de beaucoup d'intérêt de la part de la biologie moléculaire. Nous pouvons citer en particulier les travaux de D.K. Lubensky et D.R. Nelson [63] ainsi que ceux de B. Essevaz-Roulet, U. Bockelmann, et F. Heslot [31]. Ils ont inventé une nouvelle méthode pour séquencer l'ADN en "tirant" sur les brins d'ADN. Les liaisons entre les nucléotides n'ayant pas la même force, les brins vont se détacher plus ou moins vite suivant les nucléotides présents. Ceci est relié aux marches aléatoires en milieu aléatoire, les liaisons représentant l'environnement. Le problème qu'ils soulèvent est de déterminer l'environnement à partir de plusieurs réalisations de la marche. Sur ce sujet nous mentionnons également les travaux de S.Cocco, R. Monasson, et J.F. Marko [19].

Cependant, si le cas unidimensionnel est maintenant bien compris, très peu de résultats existent concernant les marches aléatoires en milieu aléatoire sur $\mathbb{Z}^d$, $d \geq 2$. Nous pouvons citer des travaux de S.A. Kalikow [51] et A.S. Sznitman et M. Zerner [93], ainsi que, dans le cas d'environnements de Dirichlet, les travaux de N. Enriquez et C. Sabot [28], de C. Sabot [84] et de C. Sabot et L. Tournier [85]. Ainsi des tentatives ont été faites dans le but d'étendre le modèle initial à des espaces différents. Nous traiterons ici plus particulièrement du cas des arbres.

Une autre direction de recherche sur les milieux aléatoires est l'extension du processus en temps continu. Une diffusion dans un potentiel Brownien a été introduite par S. Schumacher [87] et T. Brox [16]. Ce processus a des propriétés très proches de celles de la marche aléatoire en milieu aléatoire discrète, mais l'existence d'un principe d'invariance reste encore une question ouverte.

Nous commençons par un résumé des résultats existant dans le cas unidimensionnel. Cependant, comme l'objectif de cette thèse n'est pas de faire un bilan exhaustif de la recherche concernant la MAMA sur $\mathbb{Z}$, mais plutôt de décrire des extensions de ce modèle, nous prenons la liberté de ne donner que les preuves qui apportent un éclairage particulier à notre travail. Nous renvoyons le lecteur à la bibliographie, et en particulier à [99] pour des preuves plus complètes.

1.1 Définition et notations.

Soit, comme précédent, $(w_i)_{i\in\mathbb{N}}$ une famille de variables aléatoires i.i.d. à valeurs dans $[0,1]$. Nous faisons l'hypothèse (dite hypothèse d'ellipticité) qu'il existe un $\varepsilon > 0$ tel que, presque sûrement, $\varepsilon \leq w_i \leq 1-\varepsilon$. On appelle marche aléatoire dans l'environnement w la chaîne de Markov X_n, P_w définie par

$$\begin{cases} X_0 = 0, \\ P_w[X_{n+1} = x+1 | X_n = x] = w_x, \\ P_w[X_{n+1} = x-1 | X_n = x] = 1 - w_x. \end{cases}$$

Nous devons distinguer plusieurs lois de probabilité. On appellera

- μ la loi de l'environnement w,
- P_w la probabilité "quenched",
- $\mathbb{P} = \mu \otimes P_w$ la probabilité "annealed",

(les termes "quenched" et "annealed" proviennent des applications de ce modèle à la métallurgie.)
Remarque: Notons que, sous la loi annealed $\mathbb{P}$, la marche X_n n'est pas une chaîne de Markov. En effet, si la marche est à un certain point x au temps n, et que l'on sait d'autre part qu'avant l'instant n elle a fait plus de sauts de x à $x+1$ que de sauts de x à $x-1$, alors il y a de fortes chances pour que w_x soit plus proche de 1 que de 0, ainsi la marche aura plus de chances de sauter de nouveau vers $x+1$. La marche a ainsi une tendance à utiliser plusieurs fois les mêmes chemins. Pour cette raisons la MAMA est reliée à un autre modèle, dit de marches renforcées. Cela explique aussi que la MAMA aura souvent des comportements plus lents que la marche aléatoire classique.

1.2 Récurrence et Transience.

Dans cette partie nous introduisons un premier résultat, à savoir un critère de transience/récurrence. Soit $\rho_x := \frac{1-w_x}{w_x}$. les variables ρ_x, $x \in \mathbb{Z}$ sont i.i.d.. Nous appellerons ρ leur loi commune.

Théorème 1.2.1 (Récurrence/transience; Solomon 1975) *On suppose que $E_\mu[\log(\rho)]$ est bien défini, et que $P_\mu[w_0 \in \{0,1\}] = 0$. Alors*

- *Si $E_\mu[\log(\rho)] < 0$, alors $\mathbb{P}$-p.s, $X_n \to +\infty$,*
- *si $E_\mu[\log(\rho)] > 0$, alors $\mathbb{P}$-p.s, $X_n \to -\infty$,*
- *si $E_\mu[\log(\rho)] = 0$, alors $\mathbb{P}$-p.s, $\limsup X_n = +\infty$ et $\liminf X_n = -\infty$.*

Nous donnons une preuve de ce résultat, qui nous permet d'introduire des outils qui seront utiles par la suite.
Preuve : pour tout $x \in \mathbb{Z}$, soit τ_x le premier temps d'atteinte de x par X_n, et soit P_w^x la loi de la marche partant de x. Remarquons que, sous la condition d'ellipticité, pour tout $a \le x \le b$, $P_w^x(\tau_a \wedge \tau_b = \infty) = 0$. Nous pouvons donc définir

$$H(a,b,x) := P_w^x(\tau_a < \tau_b).$$

En utilisant la propriété de Markov, nous voyons que $H(a,b,x)$ satisfait l'équation

$$\begin{cases} H(a,b,a) = 1 \\ H(a,b,b) = 0 \\ H(a,b,x) = w_x H(a,b,x+1) + (1-w_x) H(a,b,x-1), \ si\ a < x < b. \end{cases}$$

Cette équation peut être résolue, et nous obtenons la formule importante

$$H(a,b,x) = \frac{\sum_{i=x+1}^{b} \prod_{j=x+1}^{i-1} \rho_j}{\sum_{i=x+1}^{b} \prod_{j=x+1}^{i-1} \rho_j + \sum_{i=a+1}^{x} \prod_{j=i}^{x} {\rho_j}^{-1}}. \tag{1.2.1}$$

La loi des grands nombre entraîne, $\mu-$ presque sûrement,

$$\prod_{j=x+1}^{i-1} \rho_j \sim \exp\left((i-x-1)(E_\mu[\log\rho] + o(1))\right), \qquad \text{quand } i \to \infty$$

et

$$\prod_{j=i}^{x} {\rho_j}^{-1} \sim \exp -\left((x-i+1)(E_\mu[\log\rho] + o(1))\right), \qquad \text{quand } i \to \infty.$$

Ainsi, dans le cas $E_\mu[\log(\rho)] < 0$, pour tout $a < x$, on obtient

$$\lim_{k\to-\infty} \lim_{n\to\infty} H(k,n,0) = 0,$$

et

$$\lim_{n\to\infty} H(-1,n,0) < 1.$$

Ceci implique que, P_w presque sûrement, $X_n \to \infty$. Le cas $E_\mu[\log(\rho)] > 0$ est direct par symétrie.

D'autre part, si $E[\log\rho_0]$ alors

$$\limsup_{i\to\infty} \sum_{j=x+1}^{i-1} \log\rho_j = \infty$$

et

$$\limsup_{i\to-\infty} \sum_{j=i}^{x} \log({\rho_j}^{-1}) = \infty,$$

On obtient donc que pour tout $k < 0$

$$\lim_{m\to\infty} H(k,m,0) = 1$$

et pour tout $m > 0$

$$\lim_{k\to-\infty} H(k,m,0) = 0.$$

Ceci implique le résultat.$\square$

1.3 Le comportement asymptotique.

1.3.1 Le régime ballistique

Le premier problème qui se pose, une fois la question de la récurrence/transience résolue, est celle de l'existence d'une loi des grands nombres, ou, autrement dit, d'une vitesse positive. En effet, il existe de nombreux exemples de processus transients qui ont des comportements "ballistiques". Le lemme suivant, du à H. Kesten [56], donne un éclairage particulier sur ce phénomène.

Lemme 1.3.1 (Kesten 1975) *Soit* $(Y_j)_{j\geq 1}$ *une suite stationnaire, alors, avec probabilité* 1*, l'évènement* $\{\lim_{n\to\infty}\sum_{i=1}^n Y_i = \infty\}$ *implique* $\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^n Y_i > 0$.

Cependant dans notre cas la suite $\{X_{n+1} - X_n\}$ n'est pas stationnaire, donc la transience n'implique pas nécessairement l'existence d'une vitesse positive. En effet, le théorème suivant, prouvé par F. Solomon [90] fait apparaître un régime intermédiaire.

Théorème 1.3.1 (Solomon 1975) *Sous les hypothèses précédentes,*

- *si* $E_\mu[\rho] < 1$, *alors* $\frac{X_n}{n} \to \frac{1-E[\rho]}{1+E[\rho]}$, $\mathbb{P}-p.s.$,
- *si* $E_\mu[\rho] < 1$, *alors* $\frac{X_n}{n} \to \frac{1-E[\rho^{-1}]}{1+E[\rho^{-1}]}$, $\mathbb{P}-p.s.$,
- *si* $1/E_\mu[\rho] \leq 1 \leq E_\mu[\rho^{-1}]$, *alors* $\frac{X_n}{n} \to 0$, $\mathbb{P}-p.s.$.

Nous donnons une idée de la preuve de ce résultat.

Preuve :

Nous rappelons que τ_1 est le premier temps d'atteinte de 1 par X_n. Soit $\theta^x w$ l'environnement translaté, défini par $\theta^x w_y = w(y+x)$.

En décomposant par rapport au premier pas, on remarque que

$$\tau_1 = \mathbb{1}_{(X_1=1)} + (1+\tau_0' + \tau_1'')\mathbb{1}_{(X_1=-1)},$$

où τ_0' est le premier temps d'atteinte de 0 par la marche partant de -1 et τ_1'' est le temps nécessaire à la marche pour atteindre 1 après son second passage en 0. En utilisant la propriété de Markov, on obtient que, sous P_w, τ_1'' a la même loi que τ_1, et que la loi de τ_0' sous P_w coïncide avec celle de τ_1 sous $P_{\theta^{-1}w}$. On obtient donc

$$E_w(\tau_1) = 1 + (1-w_0)(E_w(\tau_1) + E_{\theta^{-1}w}(\tau_1)),$$

d'où

$$E_w(\tau_1) = \frac{1}{w_0} + \rho_0 E_{\theta^{-1}w}(\tau_1).$$

En itérant cette relation, on obtient

$$E_w(\tau_1) = \frac{1}{w_0} + \frac{\rho_0}{w_1} + \frac{\rho_0\rho_{-1}}{w_2} + \cdots + \frac{\prod_{i=0}^{-(m-1}\rho_i}{w_{-m} + \prod_{i=0}^{-(m-1)}\rho_i} E_{\theta^{-m}w}(\tau_1). \tag{1.3.1}$$

En prenant l'espérance par rapport à μ, et la limite pour $m \to \infty$, on obtient

$$\mathbb{E}(\tau_1) = E_\mu\left[\sum_{i=1}^{\infty}\frac{1}{w_{-i}}\prod_{j=0}^{i-1}\rho_{-j} + \frac{1}{w_0}\right] = \frac{E_\mu(\frac{1}{w_0})}{1 - E_\mu(\rho_0)}.$$

On appelle $H_i := \tau_i - \tau_{i-1}$. On peut montrer que, sous $\mathbb{P}$, les H_i forment une suite mélangeante, ce qui signifie que pour i et j suffisamment éloignés, H_i et H_j se comportent presque comme des variables indépendantes. En particulier il est possible d'appliquer la loi des grands nombres à $\tau_n = \sum_{i=1}^n H_i$ et d'obtenir

$$\lim_{n\to\infty}\frac{\tau_n}{n} = \frac{E_\mu(\frac{1}{w_0})}{1 - E_\mu(\rho_0)}.$$

Il est ensuite facile d'en déduire que $\frac{X_n}{n}$ converge vers la limite indiquée.□

Le régime intermédiaire entre transience et ballisticité a été étudié par H. Kesten, M. V. Koslov et F. Spitzer [58]. On suppose qu'il existe un $\kappa > 0$ tel que $E_\mu\left[(\rho_0)^\kappa\right] = 1$, $E_\mu\left[(\rho_0)^\kappa \log^+\rho_0\right] < \infty$. On suppose également que l distribution de $\log\rho_0$ est non-lattice.

Théorème 1.3.2 (Kesten Koslov Spitzer 1975) *Sous les hypothèses précédentes,*

- *si* $0 < \kappa < 1$ *alors* $\frac{X_n}{n^\kappa}$ *converge vers une loi non-dégénérée explicite,*
- *si* $\kappa = 1$, *alors* $\frac{X_n \log n}{n}$ *converge vers une loi non dégénérée explicite.*

Idée de la preuve : le point essentiel est d'étudier la quantité dans (1.3.1). En effet H. Kesten [55] a montré, dans le contexte de la théorie du renouvellement pour des produits de matrices, que, sous les hypothèses du théorème, il existe une constante C_K telle que $P_\mu(\sum_{k>0}\prod_{i=-k}^{0}\rho_i > x) \sim_{x\to\infty} \frac{C_K}{x^\kappa}$. Ainsi le processus H_i peut être comparé à un processus stable d'ordre κ. Kesten, Koslov et Spitzer ont donc montré que, sous $\mathbb{P}$, $\frac{\tau_n}{n^{\frac{1}{\kappa}}}$ converge vers une loi stable complètement asymétrique d'ordre κ, multiplié par une constante. On peut en déduire facilement le résultat pour X_n. □

Récemment N. Enriquez, C. Sabot et O. Zindy [29] ont approfondi ce résultat, en décrivant explicitement la constante intervenant à la limite. Nous mentionnons également une interprétation probabiliste de la constante C_K [30].

1.3.2 Le régime lent

Nous présentons maintenant les résultats dans le cas récurrent. Il est bien connu que, pour la marche aléatoire classique, ce cas correspond à un équivalent en $\sqrt{n}$. Comme nous l'avons dit plus tôt, la marche aléatoire en milieu aléatoire a tendance à avoir un comportement plus lent. En effet, dans le cas récurrent, Ya. G. Sinaï a identifié un comportement en $(\log n)^2$.

Théorème 1.3.3 (Sinai 1982) *On suppose* $E_\mu[\log(\frac{1-w_0}{w_0})] = 0$, $\delta < w_0 < 1-\delta$, $\mu-p.s.$ *pour un* $\delta > 0$ *et* $E_\mu[(\log(\frac{1-w_0}{w_0}))^2] < \infty$. *Alors* $\frac{X_n}{(\log(n))^2}$ *converge en loi vers une distribution non dégénérée.*

Idée de la preuve: La preuve du théorème de Sinaï est l'occasion d'introduire la notion de potentiel. On considère la fonctionnelle de l'environnement W^n définie par

$$W^n(t) = \frac{\operatorname{sgn}(t)}{\log n} \sum_{i=0}^{\lfloor (\log n)^2 t \rfloor} \log \rho_i.$$

Le principe d'invariance de Donsker entraîne que, lorsque n tend vers l'infini, $W^n(t)$ converge vers un mouvement brownien (à une constante près). Nous appellerons W^n potentiel. Une explication heuristique de ce terme est contenue dans la formule (1.2.1), qui indique que la marche va visiter avec une plus grande probabilité les endroits où W^n est petit.

On appellera vallée du potentiel W^n un triplet (a, b, c) tel que $a < b < c$ et

$$W^n(b) = \min_{a \le t \le c} W^n(t) \tag{1.3.2}$$

$$W^n(a) = \max_{a \le t \le b} W^n(t) \tag{1.3.3}$$

$$W^n(c) = \max_{b \le t \le c} W^n(t) \tag{1.3.4}$$

et on désignera par profondeur de la vallée (a, b, c) la grandeur

$$d(a, b, c) := \min(W^n(a) - W^n(b), W^n(c) - W^n(b)).$$

Remarquons que si d et e vérifient $a < d < e < b$ et

$$W^n(e) - W^n(d) = \max_{a \le x < y \le b} W^n(x) - w^n(y),$$

alors (a, d, e) et (e, b, c) sont également des vallées. On dira que (a, d, e) et (e, b, c) sont des raffinements à gauche de (a, b, c). On peut définir de façon similaire des raffinements à droite.

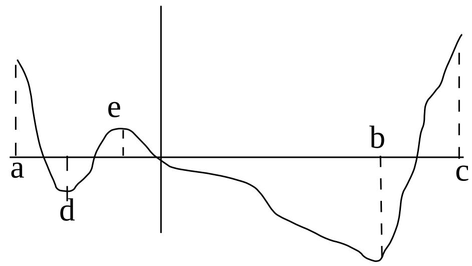


Figure 1.1: un raffinement à gauche

On définit une vallée (a_0^n, b_0^n, c_0^n) par

$$\tilde{a}_0^n = \sup\{t < 0;\ W^n(t) \geq 1\} \tag{1.3.5}$$

$$\tilde{c}_0^n = \inf\{t > 0;\ W^n(t) \geq 1\} \tag{1.3.6}$$

$$\tilde{b}_0^n = \inf\{t > \tilde{a}_0^n;\ W^n(t) = \inf_{\tilde{a}_0^n \leq t \leq \tilde{c}_0^n} W^n(t)\} \tag{1.3.7}$$

En itérant le procédé de raffinement ci-dessous à la vallée $(\tilde{a}_0^n, \tilde{b}_0^n, \tilde{c}_O^n)$, on trouve la plus petite vallée $(\tilde{a}^n, \tilde{b}^n, \tilde{c}^n)$ telle que $\tilde{a}^n < 0 < \tilde{c}^n$ et $d(\tilde{a}^n, \tilde{b}^n, \tilde{c}^n) \geq 1$. On peut montrer que b^n converge vers la position du fond de la plus petite vallée comme ci-dessus associé au mouvement brownien limite. La loi de cette position a été décrite par H. Kesten [57]. Nous allons montrer que pour tout $\eta > 0$,

$$\mathbb{P}\left(\left|\frac{X_n}{(\log n)^2} - b^n\right| > \eta\right) \rightarrow_{n\to\infty} 0.$$

On peut supposer, par exemple, que $\tilde{b}^n > 0$. Soit $a^n = \tilde{a}^n(\log n)^2$, $b^n = \tilde{b}^n(\log n)^2$ et $c^n = \tilde{c}^n(\log n)^2$. On appelle

$$T_{b,n} = \inf\{t > 0, X_t \in \{a^n, b^n\}\}.$$

La formule (1.2.1) implique facilement que, avec une probabilité qui tend vers 1, $T_{b,n}$ sera égal à τ_b. D'autre part, en utilisant la même méthode que dans la preuve de (1.3.1), on peut vérifier que, avec une probabilité qui tend vers un, $T_{b,n} < n$. Donc la marche atteint b^n avant le temps n. D'autre part, toujours en utilisant les mêmes formules, il est possible de montrer que, avec une grande probabilité, la marche va mettre longtemps à quitter la vallée, ce qui signifie que, une fois qu'elle a atteint b^n, la probabilité pour que, avant le temps n elle soit sortie de l'intervalle $[b^n - \eta(\log n)^2, b^n + \eta(\log n)^2]$ tend vers 0. Ceci implique le résultat. □

La marche aléatoire en milieu aléatoire est maintenant un modèle bien connu, mais qui fait encore l'objet de beaucoup d'intérêt [37, 20, 21, 22]. Notre but n'est pas de décrire les résultats les plus récents sur ce modèle, mais plutôt d'étudier des extensions. Nous allons tout d'abord donner une brève introduction à des processus reliés, avant de nous tourner vers l'objet principal de notre étude.

1.4 Deux extensions du modèle : la MAMA sur $\mathbb{Z}^d$ et la marche renforcée.

1.4.1 La MAMA sur $\mathbb{Z}^d$

La définition de la marche aléatoire en milieu aléatoire sur $\mathbb{Z}^d$, $d > 1$ est similaire à celle sur $\mathbb{Z}$.

On se donne une famille *i.i.d.* de vecteurs $(w(x, x+e),\ e \in \mathbb{Z}^d,\ |e| = 1)_{x\in\mathbb{Z}^d}$ telle que, presque sûrement, $\sum_{|e|=1} w(x, x+e) = 1$, et on considère la chaîne de Markov (X_n, P_w) définie par

$$\begin{cases} X_0 = 0, \\ P_w[X_{n+1} = x + e | X_n = x] = w(x, x+e). \end{cases}$$

Soit $\mathbb{P}$ la probabilité "annealed", définie comme précédemment. Malheureusement, si la définition est très simple, il est très difficile d'étendre les résultats de la dimension un au modèle multidimensionnel. Par exemple la simple question de la transience et récurrence reste ouverte.

Le modèle a été introduit par S.A. Kalikow [51], qui a également obtenu des premiers résultats.

On suppose comme précédemment une condition d'ellipticité, à savoir qu'il existe un $\epsilon > 0$ tel que $w(x, e) > \epsilon$ presque sûrement pour tout $x \in \mathbb{Z}^d$, $|e| = 1$.

Le premier résultat de Kalikow est une loi du 0-1.

Théorème 1.4.1 (Kalikow 1981) *Pour tout $l \in \mathbb{R}^d \backslash \{0\}$,*

$$\mathbb{P}(\lim_{n\to\infty} X_n.l \in \{-\infty, \infty\}) \in \{0, 1\}.$$

Il est cependant très difficile de savoir si la quantité ci-dessus vaut 0 ou 1. Cependant il existe une condition qui implique $\mathbb{P}(\lim_{n\to\infty} X_n.l \in \{-\infty, \infty\}) = 1$. Cette condition est connue comme la condition de Kalikow. Afin de l'exprimer nous devons tout d'abord introduire une chaîne de Markov auxiliaire.

On se donne un ensemble $U \subset \mathbb{Z}^d$, et on appelle T_U le premier temps de sortie de U. Soit (X_n, P^U) la chaîne de Markov de matrice de transition

$$P^U(x, x+e) = \frac{\mathbb{E}\left[E_w\left[\sum_{i=0}^{T^U} \mathbb{1}_{X_i=x}\right] w(x, x+e)\right]}{\mathbb{E}\left[E_w\left[\sum_{i=0}^{T^U} \mathbb{1}_{X_i=x}\right]\right]}.$$

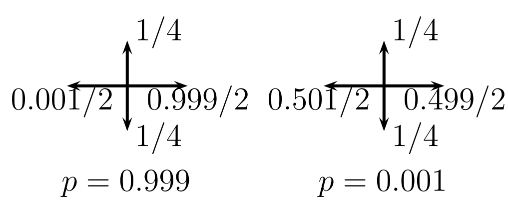
Ainsi la probabilité que la marche auxiliaire saute de x à $x+e$ est l'espérance du nombre de sauts de x à $x+e$ avant sa sortie de U, divisée par l'espérance du nombre total de passages en x.

L'intérêt de cette marche auxiliaire est que c'est une chaîne de Markov, tandis que, comme nous en faisions la remarque plus tôt, la marche originale, sous la loi annealed, n'est pas markovienne, et que, d'autre part, on peut vérifier que dès que $P^U(T_U < \infty) = 1$, on a $\mathbb{P}(T_U < \infty) = 1$ et de plus X_{T_U} a la même loi sous $\mathbb{P}$ et P^U.

Nous pouvons maintenant exprimer la condition de Kalikow relative à un vecteur l.

$$\exists \varepsilon, \inf_{U, x \in U} \sum_{|e|=1} l.eP^U(x, x+e) \geq \varepsilon. \tag{1.4.1}$$

Kalikow a montré que cette condition impliquait la transience. Cependant, la réciproque est fausse, et, bien entendu, cette condition est très difficile à vérifier en tout généralité, même si beaucoup de progrès ont été fait sur ce point récemment (voir [92, 13]). En utilisant cette méthode, Kalikow a pu montrer que, par exemple, lorsque l'environnement a la configuration suivante, alors la marche est transiente vers la droite, ce qui semble évident, mais est en fait très difficile à montrer.



Nous allons maintenant évoquer la loi des grands nombres. Pour cela nous devons introduire la notion de temps de régénération. Soit

$$\tau_1 = \inf\{n > 0; X_t.l \geq X_n.l > X_n.s, \forall s < n < t\}.$$

C'est à dire que τ_1 est le premier instant où la marche atteint son maximum dans la direction l et ne revient jamais en arrière après. Soit $D := \inf\{n \geq 0, X_n.l < X_0.l\}$.

Théorème 1.4.2 (Sznitman et Zerner 1999) *Sous les hypothèses précédentes, pourvu qu'il existe un* l *tel que* $\mathbb{P}(\lim_{n\to\infty} X_n.l = \infty\}) = 1$, *et* $\mathbb{E}[\tau_1|D = \infty]$, *alors*

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{\mathbb{E}[X_{\tau_1}|D = \infty]}{\mathbb{E}[\tau_1|D = \infty]}.$$

De plus, la condition de Kalikow relative à l *implique que* $\mathbb{P}(\lim_{n\to\infty} X_n.l = \infty\}) = 1$, *et* $\mathbb{E}[\tau_1|D = \infty] < \infty$.

Une première version de ce résultat peut être trouvée dans [93]. De nombreuses améliorations de ce résultat initial ont été obtenues depuis, voir par exemple [100, 20, 59, 23].

Nous finissons cette section en mentionnant qu'un théorème central limite a été prouvé par A.S. Sznitman [91], sous la condition de Kalikow. Nous mentionnons aussi [60] pour des résultats dans un environnement équilibré.

1.4.2 Marches aléatoires renforcées.

Nous avons fait la remarque au début de cette thèse que, dans un certain sens, la marche aléatoire en milieu aléatoire avait une tendance à "boucler" sur les mêmes trajectoires. La marche aléatoire renforcée est un modèle qui possède la même propriété. Le premier modèle de marche renforcé a été introduit par D.Coppersmith et P. Diaconis [24]. Nous pouvons la décrire de la façon suivante.

Soit un graphe G, et x un sommet de G. À chaque arête $e \in E(G)$ est associé un poids $w_0(e)$. La marche renforcée, ou marche de Diaconis partant de x est définie par

- $X_0 = x$
- à chaque pas, on choisit une arête e parmi toutes les arêtes partant de X_n, avec probabilité $P(e^*$ choisie $) = \frac{w_n(e^*)}{\sum w_n(e)}$, où la somme est prise sur toutes les arêtes partant de X_n. La marche traverse alors l'arête choisie, et au temps $n+1$ le poids de l'arête choisie est fixé à $w_{n+1}(e) = w_n(e) + 1$, tandis que toutes les autres arêtes gardent le même poids.

Nous allons maintenant supposer que $G = \mathbb{Z}^d$. Soit $(e_i)_{i\in[1,2d]}$ l'ensemble des points de $\mathbb{Z}^d$ de module 1.

Nous allons considérer une classe particulière de marches aléatoires en milieu aléatoire, à savoir les marches aléatoires dans un environnement de Dirichlet, défini comme la mesure sur $\{(x_1, ...x_{2d}) \in (0,1]^{2d}; \sum_{i=0}^{2d} x_i = 1\}$

$$\frac{\Gamma(a_1 + \cdots + a_{2d})}{\Gamma(a_1) \ldots \Gamma(a_{2d})} x_1^{a_1 - 1} \ldots x_{2d}^{a_{2d}-1} dx_1 \ldots dx_{2d},$$

où $(a_1, \ldots a_{2d})$ sont des paramètres positifs et $x_i = w(0, e_i)$.

N. Enriquez et C. Sabot [27] ont montré

Théorème 1.4.3 (Enriquez et Sabot 2002) *La loi (annealed) de la marche aléatoire sur $\mathbb{Z}^d$, partant de 0 sur un environnement de Dirichlet de paramètres $(a_1, \ldots a_{2d})$, coïncide avec la loi de la marche renforcée, dont les poids sont initialement fixés à $w_0(x, x + e_i) = a_i$.*

Pour comprendre les raisons de ce résultat, remarquons que les poids des arêtes issues d'un sommet $x \in \mathbb{Z}^d$ après n passages en x suivent la loi d'une urne de Polya, contenant $2d$ couleurs, et avec $w_0(x; x + e_i)$ boules de la $i-$ème couleur initialement dans l'urne. Il est connu, voir par exemple [36], que la suite de couleurs choisies a la même loi qu'une suite de couleurs choisies indépendamment, suivant une distribution elle-même initialement tirée au hasard suivant la loi de Dirichlet.

De nombreux résultats sont connus sur ce processus, par exemple B. Davis [26] a montré qu'en dimension 1, si tous les poids sont initialement égaux à un, la marche est soit récurrente, soit ne visite qu'un nombre fini de points, et dans ce cas elle finit par ne plus visiter que deux points. Sur un modèle relié de marche renforcée par points (au lieu de renforcée par arêtes), nous citons les travaux de R. Pemantle [76], R. Pemantle et S. Volkov [77], et finalement le récent résultat de P. Tarrès [95], à savoir que la marche renforcée par points reste finalement coincée sur cinq points.

D'autre part, dans le cas multidimensionnel, N. Enriquez et C. Sabot [28] on montré une loi des grands nombres très explicite, avec des bornes sur la vitesse, qui améliore le résultat de Sznitman et Zerner dans le cas d'un environnement de Dirichlet. D'autre part C. Sabot [84] a montré que la marche de Dirichlet était transiente quelles que soient les valeurs des paramètres, dès que $d > 3$. Nous mentionnons également le travail de C. Sabot et L. Tournier [85], concernant la transience directionnelle .

Nous allons maintenant développer un peu plus deux autres exemples d'extensions du modèle initial de MAMA. Le premier est un modèle de marche aléatoire en milieu aléatoire sur des arbres. Ce modèle est d'une certaine façon plus facile à étudier que la MAMA multi-dimensionnelle, étant notamment réversible, mais recèle néanmoins de nombreuses questions ouvertes, en effet, un arbre peut être vu, informellement, comme un espace de dimension infinie. Nous traiterons ensuite d'une extension en temps continu de la MAMA, qui a des propriétés très similaires, tandis que le lien entre les deux reste assez peu explicite.

1.5 La MAMA sur les arbres.

1.5.1 Introduction

Soit T un arbre, de racine e. Deux sommets x et y seront dits connectés, noté $x \sim y$, si x est soit le père ou un des enfants de y. Pour un sommet $x \in T$, on notera $|x|$ la distance entre x et la racine e, et $e = x^0, x^1, \dots, x^{|x|}$ le plus court chemin entre la racine et x.

Pour chaque sommet $x \in T \backslash \{e\}$ on notera son père $\overleftarrow{x}$, et ses enfants $(x_1, \cdots, x_{N(x)})$, où $N(x)$ est le nombre d'enfants de x.

Soit $\omega = (\omega(x),\, x \in T\backslash\{e\})$ une suite de vecteurs définie par $\omega(x) = (\omega(x,\, y),\, y \sim x)$ telle que $\omega(x,\, y) > 0$, $\forall y \sim x$ et $\sum_{y \sim x} \omega(x,\, y) = 1$. Plutôt que de considérer $\omega(x,\, y)$ (pour $y \sim x$ et $x \in T$), il est souvent plus pratique d'étudier $A(x_i)$, $1 \leq i \leq N(x)$ définis comme

$$A(x_i) := \frac{\omega(x,\, x_i)}{\omega(x,\, \overleftarrow{x})}, \qquad 1 \leq i \leq N(x).$$

On appellera “arbre marqué” un couple (T, A), où A est une application aléatoire des sommets de T vers $\mathbb{R}_+^*$. Nous suivons la méthode générale de J. Neveu pour construire un arbre marqué aléatoire. Soit $\mathbb{T}$ l'ensemble des arbres marqués. On introduit la filtration $\mathcal{G}_n$ sur $\mathbb{T}$ définie comme

$$\mathcal{G}_n = \sigma\{N(x), A(x_i), 1 \leq i \leq n, |x| < n, x \in T\}.$$

D'après [75], étant donnée une loi q sur $\mathbb{N} \otimes \mathbb{R}_+^{*\mathbb{N}^*}$, il existe une loi de probabilité $\mathtt{MT}$ sur $\mathbb{T}$ telle que

- la distribution de la variable aléatoire $(N(e), A(e_1), A(e_2), ...)$ est q,
- sachant $\mathcal{G}_n$, les variables aléatoires $(N(x), A(x_1), A(x_2), ...)$, pour $x \in T_n$, sont indépendantes, et de loi q.

On notera $E_q,\ P_q$ l'espérance et la probabilité associées à q, et $E_{\mathtt{MT}},\ P_{\mathtt{MT}}$ celles associées à $\mathtt{MT}$ On supposera toujours que $m := E_q[N(e)] > 1$, ce qui implique que l'arbre est infini avec une probabilité positive, et on fera toujours l'hypothèse que $(N(e), A(e_1), A(e_2), ...)$ n'est pas constant. (T, A) sera appelé “l'environnement” et on appellera “marche aléatoire sur T” la chaîne de Markov $(X_n, \mathbb{P}_T)$ définie par $X_0 = e$ et

$$\forall x, y \in T,\ \mathbb{P}_T(X_{n+1} = y | X_n = x) = \omega(x, y).$$

On appelle probabilité “annealed” la probabilité $\mathbb{P}_{\mathtt{MT}} = \mathtt{MT} \otimes \mathbb{P}_T$ qui prend en compte tout l'aléa.

Exemple (Marche biaisée sur des arbres de Galton-Watson). Lorsque $A(x) \equiv \lambda$, $\forall x$, (où $0 < \lambda < \infty$ est une constante), la marche aléatoire (X_n) est dite marche λ-biaisée sur T. Elle a été étudiée par Lyons, Pemantle et Peres [69], [68], Peres et Zeitouni [78], et Ben Arous et al. [3]. En particulier, si $A(x) \equiv 1$, $\forall x$, nous obtenons la marche aléatoire simple sur T.

Ben Arous et Hammond [4] ont considéré le cas où $A_i(x)$ ne dépend ni de x ni de i, mais peut être aléatoire. Ils ont appelé la marche obtenue marche biaisée aléatoirement sur T, et prouvé que cette marche est plus régulière en un certain sens que la marche aléatoire classique .

On note, pour $x \in T$, $C_x := \prod_{e<z\leq x} A(z) := e^{-V(x)}$. On peut associer à la marche aléatoire X_n un réseau électrique avec une conductance C_x le long de l'arête $[\overleftarrow{x}, x]$, et un réseau de capacités, avec une capacité C_x le long de $[\overleftarrow{x}, x]$ (pour plus de précisions sur ces correspondances nous renvoyons le lecteur aux chapitres 2 et 3 de [70]). Nous introduisons la fonction

$$\rho(t) := E_q\Big\{ \sum_{i=1}^{N} A(e_i)^t \Big\} \in (0, \infty], \qquad t \geq 0.$$

En particulier, $\rho(0) = \mathbf{E}(N) > 1$.

1.5.2 Récurrence/transience

Contrairement à la MAMA multidimensionnelle, la question de la transience/récurrence pour la MAMA sur des arbres a été explicitement résolue par R. Lyons et R. Pemantle [66]. Le résultat est le suivant

Théorème 1.5.1 (Lyons et Pemantle 1992) *On suppose qu'il existe $0 \leq \alpha \leq 1$ tel que ρ soit finie dans un voisinage de α, $\rho(\alpha) = \inf_{0\leq t\leq 1} \rho(t) := p$ et $\rho'(\alpha) = E_q\left[\sum_{i=1}^{N(e)} A(e_i)^\alpha \log(A(e_i))\right]$ est fini. On suppose également que $\sum_{i=1}^{N(e)} A(e_i)$ n'est pas identiquement égal à 1. Alors*

1. *si $p < 1$ la MAMA est presque sûrement récurrente positive,*
2. *si $p \leq 1$ alors la MAMA est presque sûrement récurrente*
3. *si $p > 1$, alors la MAMA est presque sûrement transience, conditionellement à l'évènement $\{T \text{ est infini}\}$.*

(Par "presque sûrement" nous voulons dire " pour MT *presque tout T").*

Remark: Si $\sum_{i=1}^{N(e)} A(e_i)$ est égal à 1, on est dans deuxième cas, $|X_n|$ est une marche aléatoire simple sans biais, et donc X_n est récurrente nulle. Cependant il existe un flot, donné par $\theta(\overleftarrow{x}, x) = C_x$.

Ce théorème a été prouvé à l'origine sous l'hypothèse que la loi des $A(e_i)$ ne dépend pas de i. Cette hypothèse a été supprimée dans [33]. Voir aussi Menshikov et Petritis [73] pour une autre preuve de ce critère via la cascade de Mandelbrot.

Le Théorème 1.5.1 ne précise pas totalement le cas $p = 1$, mais ce résultat peut être amélioré, sous des hypothèses techniques supplémentaires. Soit la condition

$$(H1) : \ \forall \alpha \in [0,1], \ E_q \left[\left(\sum_{0}^{N(e)} A(e_i)^\alpha \right) \log^+ \left(\sum_{0}^{N(e)} A(e_i)^\alpha \right) \right] < \infty.$$

Dans le cas critique nous avons le résultat suivant

Proposition 1.5.1 *On suppose $p = 1$, $m > 1$ et (H1). On suppose également que*

$$\rho'(1) = E_q \left[\sum_{i=1}^{N(e)} A(e_i) \log(A(e_i)) \right]$$

est bien défini et que ρ est finie dans un voisinage de 1. Alors,

- *si $\rho'(1) < 0$, alors, conditionellement à l'évènement $\{T$ infini$\}$, la marche est presque sûrement récurrente nulle,*
- *si $\rho'(1) = 0$, et pour un $\delta > 0$,*

  $$E_q[N(e)^{1+\delta}] < \infty,$$

  alors, conditionellement à l'évènement $\{T$ infini$\}$, la marche est presque sûrement récurrente nulle,
- *si $\rho'(1) > 0$, et, pour un $\eta > 0$, $\omega(x, \overleftarrow{x}) > \eta$ alors la marche est presque sûrement récurrente positive.*

Nous présentons des preuves générales du Théorème 1.5.1 et de la Proposition 1.5.1 dans la section 3.1.

1.5.3 Le comportement asymptotique

Le cas transient (i.e., quand $\inf_{t\in[0,1]}\psi(t) > 0$ ou $p > 1$) a reçu beaucoup d'intérêt récemment. Nous présentons ici le résultat principal, prouvé par E. Aidékon [1], sous les hypothèses que $P_q(N(e) = 0) = 0$ et que les $A(e_i)_{1\le i\le N(e)}$ sont des variables aléatoires i.i.d., independantes de $N(e)$.

Theorem 1.5.2 (Aidékon 2008) *On suppose $p > 1$. Soit $q_1 := P_q(N = 1)$, et*

$$\Lambda = Leb\left\{t \in R, E_q(A(e_1)^t) \le \frac{1}{q_1}\right\},$$

où Leb est la mesure de Lebesgue, alors,

- *Si $\Lambda > 1$, la marche a une vitesse positive presque sûrement*
- *si $\Lambda < 1$ la marche a une vitesse nulle presque sûrement.*

De plus, dans le second cas,

$$\lim_{n\to\infty} \frac{\log|X_n|}{\log n} = \Lambda, \ p.s.$$

Nous mentionnons également un autre article d'E. Aidékon [2], concernant des larges déviations, ainsi que le travail de G. Ben Arous, A. Fribergh, N. Gantert et A. Hammond [3], dans le cas où l'arbre peut avoir des feuilles.

Si $\inf_{t\in[0,1]}\rho(t) < 1$, la marche (X_n) est récurrente positive pour presque tout environnement. Dans ce cas, Y. Hu et Z. Shi [42] ont montré, sous les hypothèses additionnelles que N est déterministe et que la loi de $A(e_i)$ ne dépend pas de i, que $\frac{1}{\log n}\max_{0\le k\le n}|X_k|$ converge presque sûrement vers une constante explicite.

Nous allons maintenant traiter le cas critique $\inf_{t\in[0,1]}\rho(t) = 1$. Nous introduisons la fonction $\psi := \log\rho$. On remarque que l'hypothèse $\inf_{t\in[0,1]}\rho(t) = 1$ se traduit par $\inf_{t\in[0,1]}\psi(t) = 0$. On suppose l'existence d'un $\delta > 0$ tel que

$$\rho(t) < \infty, \quad \forall\, t \in (-\delta, 1+\delta), \qquad E_q(N^{1+\delta}) < \infty. \tag{1.5.1}$$

Il faut distinguer plusieurs cas, suivant le signe de $\psi'(1) := \mathrm{e}^{-\psi(1)}\mathbf{E}\{\sum_{i=1}^N A_i \log A_i\}$.

Nous commençons par le cas $\psi'(1) \ge 0$ (et $\inf_{t\in[0,1]}\psi(t) = 0$). Il est connu que la marche est dans ce cas très lente. En particulier, Y. Hu et Z. Shi [44]) ont montré, sous les hypothèses

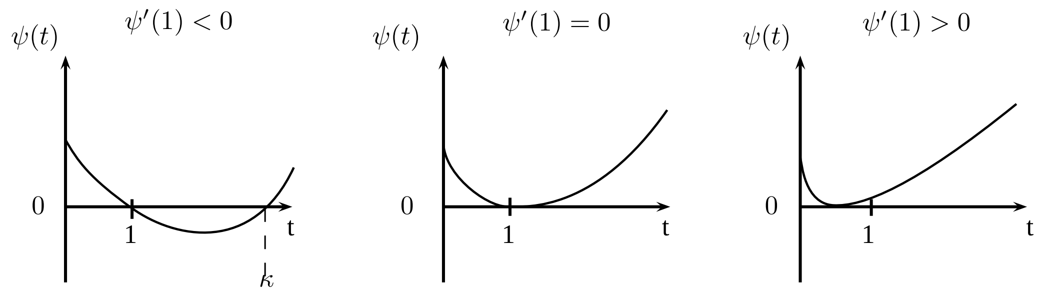


Figure 1.2: Les formes possibles pour ψ

que N est déterministe et que la loi de A_i ne dépend pas de i, qu'il existe des constantes $0 < c_1 \leq c_2 < \infty$ telles que

$$c_1 \leq \liminf_{n\to\infty} \frac{\max_{0\leq k\leq n} |X_k|}{(\log n)^3} \leq \limsup_{n\to\infty} \frac{\max_{0\leq k\leq n} |X_k|}{(\log n)^3} \leq c_2, \qquad \text{p.s.} \tag{1.5.2}$$

Nous allons montrer qu'il y a en fait convergence sûre dans (1.5.2) si $\psi'(1) \geq 0$ (et $\inf_{t\in[0,\,1]} \psi(t) = 0$). Cependant la limite dans (1.5.2), aura une nature différente suivant que $\psi'(1) = 0$ ou $\psi'(1) > 0$. Nous traitons d'abord le cas $\psi'(1) = 0$; dans ce cas la condition $\inf_{t\in[0,\,1]} \psi(t) = 0$ est équivalente à $\psi(1) = 0$.

Théorème 1.5.2 *Supposons $\psi(1) = \psi'(1) = 0$. Sur l'évènement de non-extinction,*

$$\lim_{n\to\infty} \frac{\max_{0\leq k\leq n} |X_k|}{(\log n)^3} = \frac{8}{3\pi^2\sigma^2}\,, \qquad \mathbb{P}_{\texttt{MT}}\text{-p.s.},$$

où

$$\sigma^2 := E_q\Big\{ \sum_{i=1}^{N} A(e_i)(\log A(e_i))^2 \Big\}. \tag{1.5.3}$$

Ce résultat, de même que le suivant, font partie d'un travail en collaboration avec Y. Hu et Z. Shi [35].

Nous nous tournons à présent vers le cas $\psi'(1) > 0$ (et $\inf_{t\in[0,\,1]} \psi(t) = 0$). Dans ce cas, il existe $0 < \theta < 1$ tel que

$$\psi'(\theta) = 0. \tag{1.5.4}$$

[Le précédent cas correspond donc à $\theta = 1$.]

Théorème 1.5.3 *Supposons* $\inf_{t\in[0,1]}\psi(t) = 0$ *et* $\psi'(1) > 0$. *Sur l'évènement de non-extinction,*

$$\lim_{n\to\infty} \frac{\max_{0\le k\le n}|X_k|}{(\log n)^3} = \frac{2\,\theta}{3\pi^2\psi''(\theta)}, \qquad \mathbb{P}_{\mathtt{MT}}\text{-p.s.},$$

où $\theta \in (0,\,1)$ *est tel que dans (1.5.4) and* $\psi''(\theta) = E_q\Big\{\sum_{i=1}^N A(e_i)^\theta\,(\log A(e_i))^2\Big\}$.

Les théorèmes 1.5.2 et 1.5.3 sont prouvés dans le chapitre 4 .

Lorsque $\psi'(1) < 0$ (et $\inf_{t\in[0,1]}\psi(t) = 0$), alors en définissant $\kappa := \inf\{t > 1 : \psi(t) = 0\} \in (1,\infty]$ (avec $\inf\emptyset := \infty$) et $\nu := 1 - \frac{1}{\min\{\kappa,2\}} \in (0,\,\frac{1}{2}]$, l'ordre de grandeur de $|X_n|$ est n^ν. En effet Y. Hu et Z. Shi [44] ont montré (pour $A(e_i)$ identiquement distribués et N déterministe), que

$$\lim_{n\to\infty} \frac{\log \max_{0<s<n}|X_s|}{\log n} = \nu,\ \mathbb{P}_{\mathtt{MT}} - p.s..$$

On remarque que pour $\kappa \ge 2$, la marche se comporte asymptotiquement en $n^{1/2}$. Nous allons préciser ceci par un théorème central limite. Malheureusement nous ne sommes pas dans la mesure de couvrir l'intégralité du régime $\kappa \in (2,\infty)$.

Nous commençons par quelques hypothèses techniques :

$$\exists\ 0 < \varepsilon_0 < \infty; \forall i,\ \varepsilon_0 \le A(e_i) \le \frac{1}{\varepsilon_0},\ q - p.s. \tag{1.5.5}$$

et

$$\exists N_0\ ; N(e) \le N_0,\ q - p.s. \text{ et } P_q[N \ge 2|A(e_1)] \ge \frac{1}{N_0} \tag{1.5.6}$$

Remarque : L'hypothèse (1.5.6) est en réalité plus restrictive que nécessaire, il suffit que la formule (3.3.6) soit vérifiée, et que la fonction g de la formule (3.3.9) soit bornée supérieurement et à l'écart de 0.

On remarque que ces deux hypothèses impliquent (H1).

Théorème 1.5.4 *On suppose* $N(e) \ge 1$, $q - a.s.$,*(1.5.5), (1.5.6).*

Si $p = 1$, $\rho'(1) < 0$ *et* $\kappa \in (8,\infty]$, *alors il existe une constante* $\sigma > 0$ *telle que, pour* MT *presque tout arbre* T, *le processus* $\{|X_{\lfloor nt\rfloor}|/\sqrt{\sigma^2 n}\}$ *converge en loi vers la valeur absolue d'un mouvement brownien réel issu de* 0, *lorsque* n *tend vers l'infini.*

Remarque : Ce résultat est la généralisation d'un théorème central limite obtenu par Y. Peres et O. Zeitouni [78] dans le cas d'une marche biaisée sur un arbre de Galton-Watson, où $A(x)$ est une constante égale à $\frac{1}{m}$, et donc $\kappa = \infty$. Notre preuve est largement inspirée de leurs travaux. Le cas “annealed” est plus facile, et nous pouvons réduire l'hypothèse sur κ.

Théorème 1.5.5 (F. 2009) *On suppose* $N(e) \geq 1$, $q-a.s.$, *(1.5.5), (1.5.6). Si* $p = 1$, $\rho'(1) < 0$ *et* $\kappa \in (5, \infty]$, *alors il existe une constante* $\sigma > 0$ *telle que, sous* $\mathbb{P}_{\mathtt{MT}}$, *le processus* $\{|X_{\lfloor nt \rfloor}|/\sqrt{\sigma^2 n}\}$ *converge en loi vers la valeur absolue d'un mouvement brownien réel issu de* 0, *lorsque* n *tend vers l'infini.*

Ces deux résultats sont prouvés dans le chapitre 3.

Remarque : Comme nous le verrons dans la preuve, il est possible de prouver un TCL “annealed” dès que $\kappa \in (2, \infty)$, pour une différente classe d'arbres, suivant une loi qui peut être décrite comme la distribution invariante pour le processus “vu par la particule”.

Nous présentons à présent une extension en temps continu de la marche aléatoire en milieu aléatoire

1.6 Diffusion dans un potentiel aléatoire.

Soit $(W(x))_{x\in\mathbb{R}}$ un mouvement brownien défini sur $\mathbb{R}$ partant de 0, et, pour $\kappa \in \mathbb{R}$,

$$W_\kappa(x) := W(x) - \frac{\kappa}{2}x.$$

Soit $(\beta(t))_{t\geq 0}$ un autre mouvement brownien, indépendant de W. On appellera *diffusion dans le potentiel* W_κ une solution de l'équation (formelle)

$$dX_t = d\beta_t - \frac{1}{2}W_\kappa'(X_t)dt. \tag{1.6.1}$$

W_κ' n'a bien entendu pas de sens, mais une définition mathématique de (1.6.1) peut être donnée en utilisant le générateur infinitésimal. Pour une réalisation donnée de W_κ, X_t est une diffusion réelle, issue de 0, de générateur

$$\frac{1}{2}e^{W_\kappa(x)}\frac{d}{dx}\left(e^{-W_\kappa(x)}\frac{d}{dx}\right).$$

Cette définition peut également être définie au moyen d'un changement de temps :

$$X_t = A_\kappa^{-1}\left(B(T_\kappa^{-1}(t))\right),$$

où

$$A_\kappa(x) = \int_0^x e^{W\kappa(y)} dy,$$

$$T_\kappa(t) = \int_0^t e^{-2W_\kappa(A_\kappa^{-1}(B(s)))} ds,$$

et B est un mouvement brownien standard. A_κ est la fonction d'échelle de ce processus, et sa mesure de vitesse est $2e^{-W_\kappa(x)}dx$.

Intuitivement, pour un environnement W_κ donné, la diffusion X_t a tendance à aller vers des endroits où W_κ est plus bas, et à passer beaucoup de temps dans les "vallées" de W_κ. Si l'environnement possède un "drift" κ positif, le processus sera transient vers la droite, mais il sera ralenti par ces "vallées" (voir figure 1.6). Ceci sera expliqué plus précisément dans la section 5.2.

Nous renvoyons le lecteur à [83, 82, 47] pour des généralités sur les processus de diffusion.

Nous appellerons $\mathcal{P}$ la probabilité associée à W, P_W la probabilité "quenched" associée à la diffusion, et $\mathbb{P} := \mathcal{P} \otimes P_W$ la probabilité "annealed".

T. Brox a étudié le comportement asymptotic dans le cas $\kappa = 0$. Il a montré que, sous $\mathbb{P}$,

$$\frac{X_t}{(\log t)^2} \overset{\text{(loi)}}{\to} U,$$

où U suit une loi non-dégénérée explicite.

Le cas $\kappa > 0$ a été étudié d'un coté par K. Kawazu et H. Tanaka ([53]) et d'autre part par Y. Hu, Z. Shi, M. Yor ([46]) et présente un comportement à la "Kesten-Kozlov-Spitzer": lorsque $\kappa > 1$, la diffusion a une vitesse positive; lorsque $\kappa = 1$, sous $\mathbb{P}$,

$$\frac{X_t \log t}{t} \overset{(P)}{\to} 4,$$

et, lorsque $0 < \kappa < 1$,

$$\frac{X_t}{t^\kappa} \overset{\text{(loi)}}{\to} V,$$

où V suit la loi inverse d'une loi stable complètement asymétrique d'indice κ.

Nous allons nous intéresser aux déviations entre X_t et son comportement limite, dans le cas $0 < \kappa < 1$. Notre motivation est la suivante : comme nous l'avons vu au dessus, le comportement asymptotique de la diffusion est similaire à celui de la MAMA standard. Nous allons montrer que c'est également vrai pour les déviations. Le cas discret a été étudié par A. Fribergh, N. Gantert et S. Popov [37]. Nous nous sommes inspirés de ce travail,

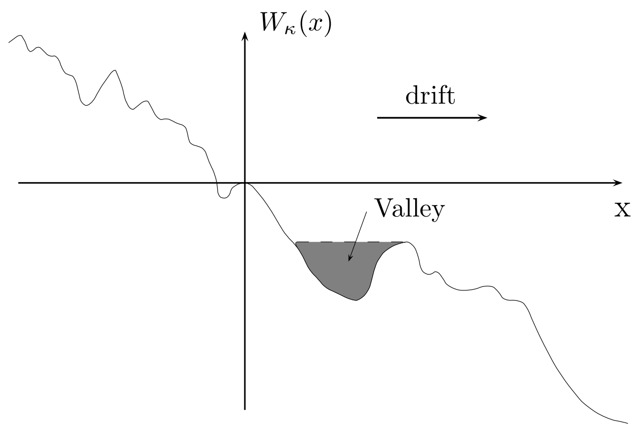


Figure 1.3: Une vallée.

en particulier pour ce qui concerne la partie “quenched slowdown”. Dans les autres cas les outils du calcul stochastique nous ont permis d'obtenir des résultats légèrement plus précis.

Nous mentionnons que ces questions ont déjà été étudiées dans les autres cas, nous renvoyons à [41] pour des estimées dans le cas $\kappa = 0$, et à [94] pour des résultats de grandes déviations dans le cas $\kappa > 1$.

Notre étude peut se séparer en quatre problèmes distincts, en effet le cas quenched et le cas annealed présentent des comportements différents, et pour chacun d'eux nous devons traiter les déviations au dessus (speedup) et en dessous (slowdown) du comportement asymptotique.

Nous commençons par le cas “annealed”. Pour u et v deux fonctions de t, on note $u \gg v$ si $u/v \to_{t\to\infty} \infty$.

Théorème 1.6.1 (Annealed speedup/slowdown) *On suppose $0 < \kappa < 1$, et $u \to \infty$ est une fonction de t telle que pour un $\varepsilon > 0$, $u \ll t^{1-\kappa-\varepsilon}$, alors il existe deux constantes C_1 et C_2 telles que*

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}\left(X_t > t^\kappa u\right)}{u^{\frac{1}{1-\kappa}}} = C_1, \tag{1.6.2}$$

et si $\log u \ll t^\kappa$,

$$\lim_{t\to\infty} u\mathbb{P}\left(X_t < \frac{t^\kappa}{u}\right) = C_2. \tag{1.6.3}$$

De plus les résultats demeurent vrais si l'on remplace X_t par $\sup_{s<t} X_s$.

Ce résultat est en fait une conséquence facile de l'étude des temps d'atteinte d'un niveau par la diffusion. Soit $H(v) = \inf\{t > 0 : X_t = v\}$. Nous avons le résultat suivant

Théorème 1.6.2 *On suppose* $0 < \kappa < 1$ *et* $\varepsilon > 0$. *Pour* $u \to \infty$, $v \to \infty$ *deux fonctions de* t *telles qu'il existe* $\varepsilon > 0$ *tel que* $u \ll v^{1-\kappa-\varepsilon}$, *in existe deux constantes positives* C_1 *et* C_2 *telles que*

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}\left[H(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right]}{u^{\frac{1}{1-\kappa}}} = C_1, \tag{1.6.4}$$

et si $\log u \ll v$,

$$\lim_{t\to\infty} u\mathbb{P}\left[H(v) > (vu)^{1/\kappa}\right] = C_2. \tag{1.6.5}$$

La preuve de ce résultat est basée sur une représentation de $H(v)$ introduite dans [41].

Nous nous tournons à présent vers le cas "quenched". Nous avons l'estimation suivante pour l'accélération

Théorème 1.6.3 (Quenched speedup) *On suppose que* $0 < \kappa < 1$, *et que* $u \to \infty$ *est une fonction de* t *telle que pour un* $\varepsilon > 0$, $u \ll t^{1-\kappa-\varepsilon}$, *alors il existe une constante positive* C_3 *telle que*

$$\lim_{t\to\infty} \frac{-\log P_W\left(X_t > t^\kappa u\right)}{u^{\frac{1}{1-\kappa}}} = C_3,\ P-p.s..$$

De plus le résultat reste vrai si l'on remplace X_t *par* $\sup_{s\le t} X_s$.

Comme précédemment, la preuve se réduit à l'étude d'asymptotiques pour les temps d'atteinte

Théorème 1.6.4 *Pour* $u \to \infty$, $v \to \infty$ *deux fonctions de* t *telles que pour un* $\varepsilon > 0$, $u \ll v^{1-\kappa-\varepsilon}$, *alors*

$$\lim_{t\to\infty} \frac{-\log P_W\left[H(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right]}{u^{\frac{1}{1-\kappa}}} = C_3,\ P-p.s.. \tag{1.6.6}$$

Pour le ralentissement, notre résultat est un peu moins précis

Théorème 1.6.5 (Quenched slowdown) *On suppose* $\kappa > 0$. *Soit* $\nu \in (0, 1\wedge\kappa)$, *alors*

$$\lim_{t\to\infty} \frac{\log(-\log P_W[H(t^\nu) > t])}{\log t} = \left(1 - \frac{\nu}{\kappa}\right) \wedge \frac{\kappa}{\kappa+1},\ P-p.s., \tag{1.6.7}$$

$$\lim_{t\to\infty} \frac{\log(-\log P_W[X_t < t^\nu])}{\log t} = \left(1 - \frac{\nu}{\kappa}\right) \wedge \frac{\kappa}{\kappa+1},\ P-p.s.. \tag{1.6.8}$$

L'ensemble des résultats de cette section est prouvé dans le chapitre 5.

1.7 Contenu.

Cette thèse est divisée en trois chapitres indépendants.

Le chapitre 3 contient la preuve dans le cas général du critère de récurrence/transience (Théorème 1.5.1), ainsi que du critère dans le cas critique (Proposition 1.5.1). A la suite de cette preuve nous présentons la preuve des Théorèmes 1.5.4 et 1.5.5. Les preuves de ces deux résultats sont assez ressemblantes, étant toutes les deux divisées en deux parties. Dans la première, nous introduirons une nouvelle distribution d'arbres, pour laquelle il sera plus facile de montrer le théorème central limite (avec un "vrai" mouvement brownien au lieu du mouvement brownien réfléchi comme processus limite). Ensuite, dans une deuxième partie, un argument de couplage nous permettra de déduire le résultat pour les arbres classiques. L'ensemble de ces résultats peuvent être retrouvés dans [33].

Le chapitre 4 contient la preuve, obtenue en collaboration avec Y. Hu et Z. Shi, des Théorèmes 1.5.2 et 1.5.3. Ces résultats reposent, entre autres, sur l'étude asymptotique de la quantité

$$\exp\Big(-\min_{|x|=n}\max_{y\in\,]\!]\varnothing,x]\!]}V(y)\Big), \tag{1.7.1}$$

dont nous verrons qu'elle est étroitement reliée à la hauteur des excursions de la marche dans l'arbre. Ces travaux ont également présentés dans [35].

Enfin, dans le chapitre 5 nous montrons les théorèmes 1.6.1, 1.6.3 et 1.6.5. Ces résultats ont été publiés dans [34].

Chapter 2

Introduction (English).

The basic motivation for studying random processes in random environment is to answer the natural question: does a process in an inhomogeneous, but regular in some sense, medium have, at least asymptotically the same behavior as a similar process in an homogeneous medium. Of course this formulation is quite unprecise, so let us first explain what we mean.

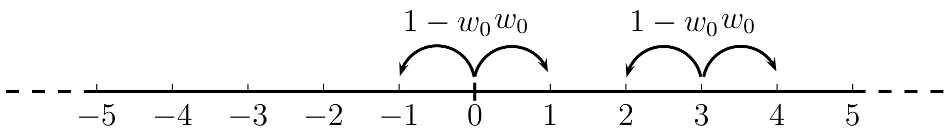
By inhomogeneous, we mean that the local configuration of the environment at some point will be random, and independent of the configuration of the environment in remote places of the space. However, it is interesting to suppose that our environment is a random perturbation of some regular environment, to translate this we will assume that some "space-shift" invariance in the distribution exists.

The first example of the study of random environment has been inspired by biophysics. Indeed, in 1967, in order to study the replication of DNA, A.A. Chernov [17] introduced a very simple model, called Random Walk in Random Environment on $\mathbb{Z}$ (RWRE), that can be described as follows.

- We consider a family of i.i.d. random variables $w_i, i \in \mathbb{N}$, such that $w_i \in [0,1]$.
- We call X_n the random walk defined as

$$\begin{cases} X_0 = 0 \\ P_w[X_{n+1} = x+1 | X_n = x] = w_x \\ P_w[X_{n+1} = x-1 | X_n = x] = 1 - w_x \end{cases}$$

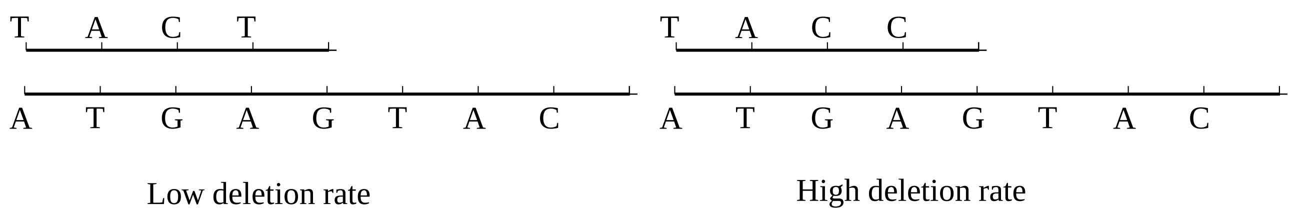
The link between this process and DNA is not obvious, so let us explicit a bit how this "environment" appears. The idea of Chernov was do model replication of DNA as follows: the



original chain of DNA is constituted of molecules A,C,G,T, and a second chain is constructed the following way.

- At the beginning the chain is empty, then one molecule forms a pair with the first molecule of the original chain

- then at each step, either another molecule is added, or the last one deleted. Of course ”wrong pairs” like C-T or A-G have a greater chance to be suppressed than the ”good pairs” A-T and G-C. Indeed if at the end there is a wrong pair in the chain, a mutation occurs.



The probabilities of formation or deletion of pairs are all different, therefore the length of the created chain is a random walk in a random environment, the environment being the original chain.

A few years later, D.E. Temkin [96], motivated by problems issued from metallurgy, made some additional work on the subject.

Many results have been obtained since on this topic, we mention in particular F. Solomon [89], Ya. G. Sinaï [88] and H. Kesten, M. V. Koslov and F. Spitzer [58].

On the other hand this process has also received much interest recently, from the part of molecular biologist. We can cite D.K. Lubensky and D.R. Nelson [63] and B. Essevaz-Roulet, U. Bockelmann, and F. Heslot [31]. They invented an interesting method for sequencing DNA by ”unzipping” DNA molecule. The links between the different pairs of nucleotides having different strength, the speed of unzipping is quite related to Random Walk in Random Environment, the DNA sequence being the ”environment”. The problem they raised is to find out a way to deduce the environment from several realization of the RWRE. On this topic we also mention the work of S.Cocco, R. Monasson, and J.F. Marko [19].

However, while the unidimensional case is now quite well understood, very few results exist for random walk in random environment on $\mathbb{Z}^d, d \geq 2$. We can cite the works of S.A. Kalikow [51] and A-S. Sznitman and M. Zerner [93] as well as the result by N. Enriquez and C. Sabot [28], C. Sabot [84] and C. Sabot et L. Tournier [85] in the context of Dirichlet environment. Therefore several attempts have been made to extend the initial model to different spaces. The case of trees will be thoroughly studied in this thesis.

An other direction for research on random medias is the extension of the latter process to continuous time. A diffusion in a Brownian potential has been introduced at first by S. Schumacher [87] and T. Brox [16], and exhibits a behavior quite similar to the behavior of one-dimensional random walk in random environment. However no direct link between the discrete and continuous time settings has ever been found. The existence of an equivalent to Donsker's invariance principle remains an open question, at the best of our knowledge.

We start with a short sum-up of the existing results in the one-dimensional case. Note that, as our aim is not to talk about random walk in random environment on $\mathbb{Z}$, but rather explain how the techniques can be used in different settings, we feel free to give only the proofs that seem relevant to our work. We refer to the bibliography, and in particular, to [99] for complete proofs.

2.1 Definition and notations.

Let, as explained in the introduction, $(w_i)_{i\in\mathbb{N}}$ be a family of i.i.d. $[0,1]-$ valued random variables. We make the"ellipticity" assumption that there exists some $\varepsilon > 0$ such that, almost surely, $\varepsilon \leq w_i \leq 1-\varepsilon$. Let X_n, P_w be the Markov chain defined as

$$\begin{cases} X_0 = 0 \\ P_w[X_{n+1} = x+1 | X_n = x] = w_x \\ P_w[X_{n+1} = x-1 | X_n = x] = 1 - w_x \end{cases}$$

We have to consider several probabilities. We call

- μ the distribution of the environment w,
- P_w the quenched probability, and
- $\mathbb{P} = \mu \otimes P_w$ the annealed probability.

(the terms quenched and annealed come from the applications of RWRE to metallurgy.)
Remark: It is interesting to note that, under $\mathbb{P}$, the walk X_n is not a Markov chain, indeed if, the walk is at some point x and if up to time n it has jumped more often from x to $x+1$ than from x to $x-1$, then it is likely that w_x is close to 1 therefore the walk has a greater probability to jump to $x+1$. Thus the walk as a tendency to stay in paths it has used before. For this reason RWRE are closely related to another model, known as reinforced walks. This will also explain that, heuristically, RWRE will be "slower" than traditional random walk.

2.2 Recurrence and Transience.

In this part we introduce, and show a first result concerning transience and recurrence. Let $\rho_x := \frac{1-w_x}{w_x}$. ρ_x, $x \in \mathbb{Z}$ are i.i.d. random variables. We call ρ their common distribution.

Theorem 2.2.1 (Recurrence/transience; Solomon 1975) *We make the assumption that $E_\mu[\log(\rho)]$ is well defined, and that $P_\mu[w_0 \in \{0,1\}] = 0$. then*

- *If $E_\mu[\log(\rho)] < 0$, then $\mathbb{P}$-a.s, $X_n \to +\infty$,*
- *if $E_\mu[\log(\rho)] > 0$, then $\mathbb{P}$-a.s, $X_n \to -\infty$,*
- *if $E_\mu[\log(\rho)] = 0$, then $\mathbb{P}$-a.s, $\limsup X_n = +\infty$ and $\liminf X_n = -\infty$.*

We give a short proof for this result, as it allows us to introduce some of the tools that will be used in the sequence.
Proof : for every $x \in \mathbb{Z}$, we call τ_x the first hitting time of x by X_n, and we denote by P_w^x the law of the walk started at x. Note that the ellipticity assumption implies that for every $a \le x \le b$, $P_w^x(\tau_a \wedge \tau_b = \infty) = 0$. Therefore we can define

$$H(a,b,x) := P_w^x(\tau_a < \tau_b).$$

Using the Markov property, we see that $H(a,b,x)$ satisfies

$$\begin{cases} H(a,b,a) = 1 \\ H(a,b,b) = 0 \\ H(a,b,x) = w_x H(a,b,x+1) + (1-w_x)H(a,b,x-1), \quad \text{for } a < x < b. \end{cases}$$

This can be easily solved, to obtain the fundamental formula

$$H(a,b,x) = \frac{\sum_{i=x+1}^{b} \prod_{j=x+1}^{i-1} \rho_j}{\sum_{i=x+1}^{b} \prod_{j=x+1}^{i-1} \rho_j + \sum_{i=a+1}^{x} \prod_{j=i}^{x} \rho_j{}^{-1}}. \tag{2.2.1}$$

Using the law of large numbers, we get that, $\mu-$ almost surely,

$$\prod_{j=x+1}^{i-1} \rho_j \sim \exp\left(i-x-1\right)(E_\mu[\log \rho] + o(1)),$$

while

$$\prod_{j=i}^{x} {\rho_j}^{-1} \sim \exp -(x-i+1)(E_\mu[\log \rho] + o(1))$$

as i goes to infinity. Therefore, in the case $E_\mu[\log(\rho)] < 0$, for all $a < x$, we get

$$\lim_{k \to -\infty} \lim_{n \to \infty} H(k, n, 0) = 0,$$

and

$$\lim_{n \to \infty} H(-1, n, 0) < 1.$$

This implies that P_w almost surely, $X_n \to \infty$. The case $E_\mu[\log(\rho)] > 0$ is direct by symmetry.

On the other hand if $E[\log \rho_0]$ then it is known that

$$\limsup_{i \to \infty} \sum_{j=x+1}^{i-1} \log \rho_j = \infty$$

and

$$\limsup_{i \to -\infty} \sum_{j=i}^{x} \log({\rho_j}^{-1}) = \infty,$$

therefore we get easily that for every $k < 0$

$$\lim_{m \to \infty} H(k, m, 0) = 1$$

and for every $m > 0$

$$\lim_{k \to -\infty} H(k, m, 0) = 0.$$

This implies the result.□

2.3 The asymptotic behavior.

2.3.1 The ballistic regime

The first question one can ask about a process, once the question of recurrence solved, is the question of the existence of a law of large number, or, said in another way, of a positive speed. Indeed, many example of transient process exhibit ballistic behaviors. We can for example cite the following lemma, due to H. Kesten [56].

Lemma 2.3.1 *For any real, stationnary sequence $(Y_j)_{j\geq 1}$, with probability 1 the event $\{\lim_{n\to\infty}\sum_{i=1}^n Y_i = \infty\}$ implies $\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^n Y_i > 0$.*

However in our case the sequence $\{X_{n+1} - X_n\}$ is not stationary, and the transience does not always imply a positive speed. Actually an intermediate behavior appears, as proved by F. Solomon.

Theorem 2.3.2 (Solomon 1975) *Under the previous assumptions,*

- *if $E_\mu[\rho] < 1$, then $\frac{X_n}{n} \to \frac{1-E[\rho]}{1+E[\rho]}$ $\mathbb{P} - a.s.$,*
- *If $E_\mu[\rho] < 1$, then $\frac{X_n}{n} \to \frac{1-E[\rho^{-1}]}{1+E[\rho^{-1}]}$ $\mathbb{P} - a.s.$,*
- *If $1/E_\mu[\rho] \leq 1 \leq E_\mu[\rho^{-1}]$, then $\frac{X_n}{n} \to 0$.*

We give a short idea of the proof of this result.

Proof : We recall that τ_1 is the first hitting time of 1 by X_n. We call $\theta^x w$ the environment described as $\theta^x w_y = w(y+x)$.

Note that

$$\tau_1 = \mathbb{1}_{(X_1=1)} + (1+\tau_0' + \tau_1'')\mathbb{1}_{(X_1=-1)},$$

where τ_0' is the first hitting time of 0 by the walk started at -1 and τ_1'' is the time it takes for the walk to reach 1 after its second hitting of 0. Using Markov's property, we get that, under P_w, τ_1'' has the same law as τ_1, and that the law of τ_0' under P_w is the same as the law of τ_1 under $P_{\theta^{-1}w}$. Therefore we obtain quite easily

$$E_w(\tau_1) = 1 + (1-w_0)(E_w(\tau_1) + E_{\theta^{-1}w}(\tau_1)),$$

hence

$$E_w(\tau_1) = \frac{1}{w_0} + \rho_0 E_{\theta^{-1}w}(\tau_1).$$

We can iterate this relation, and obtain

$$E_w(\tau_1) = \frac{1}{w_0} + \frac{\rho_0}{w_1} + \frac{\rho_0\rho_{-1}}{w_2} + \cdots + \frac{\prod_{i=0}^{-(m-1}\rho_i}{w_{-m} + \prod_{i=0}^{-(m-1)}\rho_i} E_{\theta^{-m}w}(\tau_1). \tag{2.3.1}$$

Taking the expectation under μ, and the limit as $m\to\infty$, we get

$$\mathbb{E}(\tau_1) = E_\mu\left[\sum_{i=1}^{\infty}\frac{1}{w_{-i}}\prod_{j=0}^{i-1}\rho_{-j} + \frac{1}{w_0}\right] = \frac{E_\mu(\frac{1}{w_0})}{1-E_\mu(\rho_0)}.$$

Now if we call $H_i := \tau_i - \tau_{i-1}$, under $\mathbb{P}$ one can show that the $(H_i)_{i\geq 1}$ are a strongly mixing sequence, meaning that for i and j remote enough, H_i and H_j behave almost like independent random variables. In particular we can apply the law of large numbers to $\tau_n = \sum_{i=1}^n H_i$ and get that

$$\lim_{n\to\infty} \frac{\tau_n}{n} = \frac{E_\mu(\frac{1}{w_0})}{1 - E_\mu(\rho_0)}.$$

It is then easy to check that $\frac{X_n}{n}$ converges to the desired limit. □

As we said earlier, there is an intermediate case between recurrence and ballisticity, that has been described by H. Kesten, M. V. Koslov and F. Spitzer [58]. Suppose there exists $\kappa > 0$ such that $E_\mu\left[(\rho_0)^\kappa\right] = 1$, $E_\mu\left[(\rho_0)^\kappa \log^+ \rho_0\right] < \infty$. We also assume that the distribution of $\log \rho_0$ is non-lattice.

Theorem 2.3.3 (Kesten Koslov Spitzert 1975) *Under the previous assumptions,*

- *if* $0 < \kappa < 1$ *then* $\frac{X_n}{n^\kappa}$ *converges to an explicit non-degenerate distribution,*
- *If* $\kappa = 1$, *then* $\frac{X_n \log n}{n}$ *converges to an explicit non-degenerate distribution.*

Outline of the proof : The proof of this result mainly relies on the study of the quantity expressed in (2.3.1). Indeed H. Kesten showed [55], in the context of renewal theory for product of matrices, that, under the above assumptions, for some constant C_K, $P_\mu(\sum_{k>0} \prod_{i=-k}^0 \rho_i > x) \sim_{x\to\infty} \frac{C_K}{x^\kappa}$. Therefore, the process H_i can be compared to a stable process of order κ, thus Kesten, Koslov and Spitzer proved that, under $\mathbb{P}$, $\frac{\tau_n}{n^{\frac{1}{\kappa}}}$ converges in distribution to a constant times the completely asymmetric stable law of order κ. Getting the result for X_n is then quite easy.□

This result has been improved recently by N. Enriquez, C. Sabot and O. Zindy, [29] who obtained the constant explicitly. We also mention a probabilistic interpretation of the constant C_K [30].

2.3.2 The slow regime

We now study the recurrent case. In this case, for classical random walk, the typical behavior is a $\sqrt{n}$ equivalent. As we said earlier, the random walk in random environment tends to be slower than random walk, indeed, in the recurrent case, Ya.G. Sinaï [88], found out a $(\log n)^2$ behavior.

Theorem 2.3.4 (Sinai 1982) *Suppose* $E_\mu[\log(\frac{1-w_0}{w_0})] = 0$, $\delta < w_0 < 1-\delta$, $\mu-a.s.$ *for some* $\delta > 0$ *and* $E_\mu[(\log(\frac{1-w_0}{w_0}))^2] < \infty$, *then* $\frac{X_n}{(\log(n))^2}$ *converges to some non-degenerate distribution.*

Outline of the proof : The proof of Sinaï's theorem is the opportunity to introduce the notion of potential. We consider the functional of the environment W^n define as

$$W^n(t) = \frac{\mathrm{sgn}(t)}{\log n} \sum_{i=0}^{\lfloor (\log n)^2 t \rfloor} \log \rho_i.$$

Donsker's invariance principle immediately implies that, as n goes to infinity, $W^n(t)$ converges to some constant times a Brownian motion. We will refer to this functional as a potential. An heuristic explanation for this term is the formula (2.2.1), that states that the walk will go with a greater probability to places where W^n is small.

We call a valley of the potential W^n a triple (a, b, c) such that $a < b < c$ and

$$W^n(b) = \min_{a \le t \le c} W^n(t) \tag{2.3.2}$$

$$W^n(a) = \max_{a \le t \le b} W^n(t) \tag{2.3.3}$$

$$W^n(c) = \max_{b \le t \le c} W^n(t) \tag{2.3.4}$$

and we call "depth" of the valley (a, b, c)

$$d(a, b, c) := \min(W^n(a) - W^n(b), W^n(c) - W^n(b))$$

Note that if d and e are such that $a < d < e < b$ and

$$W^n(e) - W^n(d) = \max_{a \le x < y \le b} W^n(x) - w^n(y),$$

than (a, d, e) and (e, b, c) are also valleys. We say that (a, d, e) and (e, b, c) are a left refinement of (a, b, c). We similarly define a right refinement.

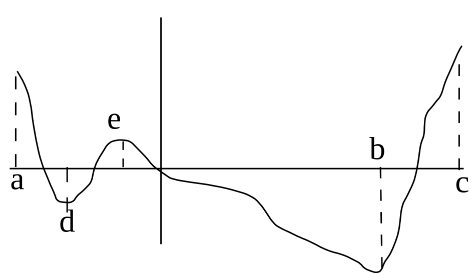


Figure 2.1: a left refinement

.

We now define a valley (a_0^n, b_0^n, c_0^n) by

$$\tilde{a}_0^n = \sup\{t < 0;\ W^n(t) \geq 1\} \tag{2.3.5}$$

$$\tilde{c}_0^n = \inf\{t > 0;\ W^n(t) \geq 1\} \tag{2.3.6}$$

$$\tilde{b}_0^n = \inf\{t > \tilde{a}_0^n;\ W^n(t) = \inf_{\tilde{a}_0^n \leq t \leq \tilde{c}_0^n} W^n(t)\} \tag{2.3.7}$$

By iterating the above refinement process to the valley $(\tilde{a}_0^n, \tilde{b}_0^n, \tilde{c}_O^n)$, we find the smallest valley $(\tilde{a}^n, \tilde{b}^n, \tilde{c}^n)$ such that $\tilde{a}^n < 0 < \tilde{c}^n$ and $d(\tilde{a}^n, \tilde{b}^n, \tilde{c}^n) \geq 1$. One can show that b^n converges to the position of the bottom of the smallest valley associated to the limiting Brownian motion. The law of this position have been studied by H. Kesten [57]. We are going to show that, for any $\eta > 0$,

$$\mathbb{P}\left(\left|\frac{X_n}{(\log n)^2} - b^n\right| > \eta\right) \to_{n\to\infty} 0.$$

Without loss of generality, we can assume that $\tilde{b}^n > 0$. Recalling the definition of W^n, we introduce $a^n = \tilde{a}^n(\log n)^2$, $b^n = \tilde{b}^n(\log n)^2$ and $c^n = \tilde{c}^n(\log n)^2$. We call

$$T_{b,n} = \inf\{t > 0, X_t \in \{a^n, b^n\}\}.$$

It is not difficult to check, using formula (2.2.1), that, with probability going to one, $T_{b,n}$ will be equal to τ_b. On the other hand, using the same method as in the proof of (2.3.1), one can check that, with probability going to one, $T_{b,n} < n$, therefore the walk will hit b^n before time n with probability going to n. On the other hand, still using the same formulae, one can show that, with high probability, the walk will take a very long time to get off the valley, meaning that, once it has hit b^n, the probability that, by time n, it has escaped from the interval $[b^n - \eta(\log n)^2, bn + \eta(\log n)^2]$ goes to zero. This implies the result□

Random walk in random environment is now quite well known, but still receives much interest [37, 20, 21, 22]. Our aim is not to describe the recent results on this topic, but rather to study several extension of this model. We first give a short introduction to two related process, before turning to our main topic.

2.4 Two extension of the model: the RWRE on $\mathbb{Z}^d$ and reinforced random walk.

2.4.1 RWRE on $\mathbb{Z}^d$

The definition of RWRE on $\mathbb{Z}^d$, $d > 1$ is similar to the definition on $\mathbb{Z}$.

We take a family of *i.i.d.* vectors $(w(x, x+e),\ e \in \mathbb{Z}^d,\ |e| = 1)_{x \in \mathbb{Z}^d}$ such that, almost surely, $\sum_{|e|=1} w(x, x+e) = 1$, and consider the Markov chain (X_n, P_w) defined as

$$\begin{cases} X_0 = 0, \\ P_w[X_{n+1} = x + e | X_n = x] = w(x, x+e). \end{cases}$$

Let $\mathbb{P}$ denote as before the annealed probability. Unfortunately, while the definition is quite simple, it is very difficult to extend the results in the one-dimension case to two ore more dimensions. For example the simple question of transience and recurrence remains open up to now.

The definition and initial work on the subject has been made by S. A. Kalikow [51]. We suppose that some ellipticity condition holds, that is, there exists some $\epsilon > 0$ such that $w(x, e) > \epsilon$ almost surely for all $x \in \mathbb{Z}^d$, $|e| = 1$.

The first result of Kalikow is a 0-1 law.

Theorem 2.4.1 (Kalikow 1981) *For all* $l \in \mathbb{R}^d \backslash \{0\}$,

$$\mathbb{P}(\lim_{n\to\infty} X_n.l \in \{-\infty, \infty\}) \in \{0, 1\}.$$

It is however rather difficult to know wether it is equal to 0 or 1. We are now going to introduce a conditions that imply $\mathbb{P}(\lim_{n\to\infty} X_n.l \in \{-\infty, \infty\} = 1$. This conditions is known as Kalikow's condition. To express it we must first introduce an auxiliary Markov chain. We take some subset $U \subset \mathbb{Z}^d$, and call T_U the exit time of U. Let (X_n, P^U) be the Markov chain with transition probabilities

$$P^U(x, x+e) = \frac{\mathbb{E}\left[E_w\left[\sum_{i=0}^{T^U} \mathbb{1}_{X_i = x}\right] w(x, x+e)\right]}{\mathbb{E}\left[E_w\left[\sum_{i=0}^{T^U} \mathbb{1}_{X_i = x}\right]\right]}.$$

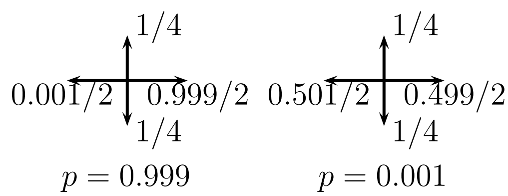
Basically the probability that the auxiliary chain goes from x to $x + e$ is the expected number of jumps of the original walk from x to $x + e$ before exiting U over the expected number of visits to x.

The interest of this auxiliary chain is that it is a Markov chain, while the original walk, as we said earlier, is not a Markov chain under the annealed law, and furthermore, one can show that, whenever $P^U(T_U < \infty) = 1$, we have $\mathbb{P}(T_U < \infty) = 1$ and furthermore X_{T_U} has the same distribution under $\mathbb{P}$ and P^U.

We are now able to state Kalikow's condition relative to some vector l.

$$\exists \varepsilon,\ \inf_{U, x \in U} \sum_{|e|=1} l.eP^U(x, x+e) \geq \varepsilon. \tag{2.4.1}$$

Kalikow showed that this condition implied transience, however, the converse is not true, and, of course, this condition is very difficult to check in generality, but a lot of improvement has been done recently (see [92, 13]) Using this method, Kalikow managed to show that, for example, if the distribution of the environment is the following, then the walk is transient



to the right, which seems quite obvious, but is actually very difficult to prove.

We are now going to present a law of large numbers. Toward this goal, we have to introduce regeneration times. Let

$$\tau_1 = \inf\{n > 0; X_t.l \geq X_n.l > X_n.s, \forall s < n < t\}.$$

That is, τ_1 is the first instant where the walk hits its maximum in direction l and never gets back after. We also call $D := \inf\{n \geq 0, X_n.l < X_0.l\}$.

Theorem 2.4.2 (Sznitman Zerner 1999) *Under the previous assumptions, provided that for some l, $\mathbb{P}(\lim_{n\to\infty} X_n.l = \infty\}) = 1$, and $\mathbb{E}[\tau_1 | D = \infty]$, then*

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{\mathbb{E}[X_{\tau_1} | D = \infty]}{\mathbb{E}[\tau_1 | D = \infty]}.$$

Moreover, Kalikow's condition relative to l implies that $\mathbb{P}(\lim_{n\to\infty} X_n.l = \infty\}) = 1$, and $\mathbb{E}[\tau_1 | D = \infty] < \infty$.

A first version of this result can be found in [93]. Several improvement of this initial result have been done since, see for example [100, 20, 59, 23].

We finish this part by mentioning that a central limit theorems have been proven under Kalikow's condition by A.S. Sznitman [91]. We also mention [60] for results in a balanced environment.

2.4.2 Reinforced random walk.

We remarked earlier that, in some sense, a random walk in random environment has a tendency to stay in the same paths. Another model that has the same property is called reinforced random walk. The initial model of reinforced random walk has been introduced by D. Coppersmith and P. Diaconis [24], and can be described the following way. Let G be a graph, and x a vertex in G. To each edge $e \in E(G)$ is associated a weight $w_0(e)$. Then the reinforced random walk, or Diaconis random walk started at x is defined by

- $X_0 = x$
- at each step, we chose one edge e amongst all the edges starting from X_n, with probability $P(e^* \text{ is chosen }) = \frac{w_n(e^*)}{\sum w_n(e)}$, where the sum is taken over all the edges issued from X_n. The walker then crosses the chosen edge, and the weight at time $n+1$ of the chosen edge is defined as $w_{n+1}(e) = w_n(e) + 1$, while all the remaining edge keep the same weight.

From now on we will assume that $G = \mathbb{Z}^d$. We consider some ordering $(e_i)_{i \in [1,2d]}$ of the vertices in $\mathbb{Z}^d$ with modulus 1.

We shall consider a particular class of random walks in random environment, namely RWRE with environment following a Dirichlet distribution, defined as the measure on $\{(x_1, ...x_{2d}) \in (0,1]^{2d}; \sum_{i=0}^{2d} x_i = 1\}$

$$\frac{\Gamma(a_1 + \cdots + a_{2d})}{\Gamma(a_1) \ldots \Gamma(a_{2d})} x_1^{a_1 - 1} \ldots x_{2d}^{a_{2d}-1} dx_1 \ldots dx_{2d},$$

where $(a_1, \ldots a_{2d})$ are positive parameters and $x_i = w(0, e_i)$.

N. Enriquez and C. Sabot [27] showed the following

Theorem 2.4.3 (Enriquez Sabot 2002) *The (annealed) distribution of the RWRE on $\mathbb{Z}^d$, started at 0 on a Dirichlet environment with parameters $(a_1, \ldots a_{2d})$, coincides to the distribution of the reinforced random walk with initial weights set at $w_0(x, x + e_i) = a_i$.*

To understand the reasons for this result, note that the weights of the edges issued from any vertex $x \in \mathbb{Z}^d$ after n passages to x follow the law of a Polya's urn, with $2d$ colors, and with $w_0(x; x + e_i)$ balls of the i-th color initially in the urn. It is known, see for example [36] that the sequence of the colors picked in the urn has the same law as a sequence of colors picked independently of each other but with the same distribution, initially randomly chose following Dirichlet's distribution.

A lot of results are known on this process. For example B. Davis [26] showed that, on $\mathbb{Z}$, if the initial weights are all set to 1, then the reinforced random walk is either recurrent, or has finite range, and in this case it finally gets stuck in two points. On a similar model of vertex reinforced random walk, we mention the works of R. Pemantle, [76], R. Pemantle and S. Volkov [77], and finally the recent result by P. Tarrès [95], that a vertex reinforced random finally gets stuck in five points.

On the other hand, for the multidimensional RWRE setting we mention the works by N. Enriquez and C. Sabot [28], concerning a quite explicit law of large numbers, with bounds on the speed, improving Sznitman and Zerner's law of large numbers in the case of a Dirichlet environment. On the other hand C. Sabot [84] has shown that Dirichlet's random walk is transient whenever $d \geq 3$. We also mention the work of C. Sabot and L. Tournier [85] on directional transience.

We are now going to develop a bit more two other examples of extensions of the initial model of RWRE. The first model we present is a model of RWRE on trees. This model turns out to be slightly easier to deal with than the multidimensional RWRE, but still keeps some unanswered questions. The second one is a time continuous extension of RWRE, which happens to have properties very similar to the standard model, while the link between both is somewhat unclear.

2.5 The RWRE on trees.

2.5.1 Introduction

Let T be a tree rooted at e. Two vertices x and y are said to be connected, and denoted by $x \sim y$, if x is either the parent or a child of y. For a vertex $x \in T$, we denote by $|x|$ the distance between x and the root e, and $e = x^0, x^1, \ldots, x^{|x|}$ the shortest path between the root and x.

For each vertex $x \in T \backslash \{e\}$, we denote its parent by $\overleftarrow{x}$, and its children by $(x_1, \cdots, x_{N(x)})$, where $N(x)$ stands for the number of the children of x.

Let $\omega = (\omega(x), x \in T\backslash\{e\})$ be a sequence of vectors defined by $\omega(x) = (\omega(x, y), y \sim x)$ such that $\omega(x, y) > 0, \forall y \sim x$ and that $\sum_{y\sim x} \omega(x, y) = 1$. Instead of looking at $\omega(x, y)$ (for $y \sim x$ and $x \in T$), it is often more convenient to study $A(x_i),\ 1 \leq i \leq N(x)$ defined by

$$A(x_i) := \frac{\omega(x, x_i)}{\omega(x, \overleftarrow{x})}, \qquad 1 \leq i \leq N(x).$$

We call marked tree a couple (T, A), where A is a random application from the vertices of T to $\mathbb{R}_+^*$. We give a quite general method to construct a random marked tree. Let $\mathbb{T}$ be the set of marked trees. We introduce the filtration $\mathcal{G}_n$ on $\mathbb{T}$ defined as

$$\mathcal{G}_n = \sigma\{N(x), A(x_i), 1 \leq i \leq n, |x| < n, x \in T\}.$$

Following [75], given a probability measure q on $\mathbb{N} \otimes \mathbb{R}_+^{*\mathbb{N}^*}$, there exists a probability measure $\mathtt{MT}$ on $\mathbb{T}$ such that

- the distribution of the random variable $(N(e), A(e_1), A(e_2), ...)$ is q,
- given $\mathcal{G}_n$, the random variables $(N(x), A(x_1), A(x_2), ...)$, for $x \in T_n$, are independent and their conditional distribution is q.

We will use the notations E_q, P_q for the expectation and probability associated to q; and $E_{\mathtt{MT}}$, $P_{\mathtt{MT}}$ for the expectation and probability associated to $\mathtt{MT}$. We will always assume $m := E_q[N(e)] > 1$, ensuring that the tree is infinite with a positive probability, and we will always assume that $(N(e), A(e_1), A(e_2), ...)$ is not a constant vector.

(T, A) will be called "the environment", and we call "random walk on T" the Markov chain $(X_n, \mathbb{P}_T)$ defined by $X_0 = e$ and

$$\forall x, y \in T,\ \mathbb{P}_T(X_{n+1} = y | X_n = x) = \omega(x, y).$$

We call "annealed probability" the probability $\mathbb{P}_{\mathtt{MT}} = \mathtt{MT} \otimes \mathbb{P}_T$ taking into account the total alea.

Example (Biased random walk on a Galton–Watson tree). When $A(x) \equiv \lambda$, $\forall x$, (where $0 < \lambda < \infty$ is a constant), the random walk (X_n) is the λ-biased random walk on T studied by Lyons, Pemantle and Peres [69], [68], Peres and Zeitouni [78], and Ben Arous et al. [3]. More particularly, if $A_i(x) \equiv 1$, $\forall x$, $\forall i$, we get the simple random walk on T.

Ben Arous and Hammond [4] considered the case where $A(x)$ does not depend on x nor on i, but can be random. They called the resulting walk (X_n) randomly biased walk on T, and proved that the walk is more regular in some sense than the biased random walk. $\square$

We set, for $x \in T$, $C_x := \prod_{e<z\leq x} A(z) := e^{-V(x)}$. We can associate to the random walk X_n an electrical network with conductance C_x along $[\overleftarrow{x}, x]$, and a capacited network with capacity C_x along $[\overleftarrow{x}, x]$ (for more precisions on this correspondence we refer to the chapters 2 and 3 of [70]). Let

$$\rho(t) := E_q\Big\{\sum_{i=1}^{N} A(e_i)^t\Big\} \in (-\infty,\ \infty], \qquad t \geq 0.$$

In particular, $\rho(0) = E_q(N) > 1$.

2.5.2 Recurrence/transience

Unlike multidimensional RWRE, the question of transience/recurrence for RWRE on trees was explicitly solved by R. Lyons and R. Pemantle [66]. Their result is the following

Theorem 2.5.1 *We suppose that there exists* $0 \leq \alpha \leq 1$ *such that* ρ *is finite in a small neighborhood of* α, $\rho(\alpha) = \inf_{0 \leq t \leq 1} \rho(t) := p$ *and* $\rho'(\alpha) = E_q\left[\sum_{i=1}^{N(e)} A(e_i)^\alpha \log(A(e_i))\right]$ *is finite. We assume that* $\sum_{i=1}^{N(e)} A(e_i)$ *is not identically equal to* 1.

Then,

1. *if* $p < 1$ *then the RWRE is a.s. positive recurrent, the electrical network has zero conductance a.s., and the capacited network admits no flow a.s..*
2. *if* $p \leq 1$ *then the RWRE is a.s. recurrent, the electrical network has zero conductance a.s. and the capacited network admits no flow a.s..*
3. *if* $p > 1$, *then, given non-extinction, the RWRE is a.s. transient, the electrical network has positive conductance a.s. and the capacited network admits flow a.s..*

(By "almost surely" we mean "for `MT` *almost every* T*").*

Remark: In the case where $\sum_{i=1}^{N(e)} A(e_i)$ is identically equal to 1, which belongs to the second case, $|X_n|$ is a standard unbiased random walk, therefore X_n is null recurrent. However, there do exist a flow, given by $\theta(\overleftarrow{x}, x) = C_x$.

This theorem was originally proved under the additional condition that the distribution of A_i does not depend on i; this condition was released in Faraud [33]. See also Menshikov and Petritis [73] for a proof of this criterion (under the additional assumptions that $N > 1$ is deterministic and that the law of A_i does not depend on i) via Mandelbrot's multiplicative cascades.

Theorem 2.5.1 does not give a full answer in the case $p = 1$, but this result can be improved, provided some technical assumptions are fullfilled. We introduce the condition

$$(H1): \ \forall \alpha \in [0,1], \ E_q\left[\left(\sum_0^{N(e)} A(e_i)^\alpha\right) \log^+\left(\sum_0^{N(e)} A(e_i)^\alpha\right)\right] < \infty,$$

In the critical case, we have the following

Proposition 2.5.2 *We suppose* $p = 1$, $m > 1$ *and (H1). We also suppose that* $\rho'(1) = E_q\left[\sum_{i=1}^{N(e)} A(e_i) \log(A(e_i))\right]$ *is defined and that* ρ *is finite in a small neighborhood of* 1. *Then,*

- *if* $\rho'(1) < 0$, *then the walk is a.s. null recurrent, conditionally on the system's survival,*
- *if* $\rho'(1) = 0$ *and for some* $\delta > 0$,

$$E_q[N(e)^{1+\delta}] < \infty,$$

then the walk is a.s. null recurrent, conditionally on the system's survival,

- *if* $\rho'(1) > 0$, *and if for some* $\eta > 0$, $\omega(x, \overleftarrow{x}) > \eta$ *almost surely, then the walk is almost surely positive recurrent.*

We present in section 3.1 the proofs of Theorem 2.5.1 and Proposition 2.5.2 in the most general case

2.5.3 The asymptotic behavior

The transient case (i.e., if $\inf_{t\in[0,1]} \psi(t) > 0$ or equivalently $p > 1$) has received much research interest recently. We present here the main result, proved by E. Aidékon [1], under the assumptions that $P_q(N(e) = 0) = 0$ and that the $A(e_i)_{1\le i\le N(e)}$ are i.i.d. random variables, independent of $N(e)$.

Theorem 2.5.3 (Aidékon 2008) *Suppose* $p > 1$ *and let* $q_1 := P_q(N = 1)$, *and*

$$\Lambda = Leb\{t \in R, E_q(A(e_1)^t) \le \frac{1}{q_1},$$

where Leb stands for the Lebesgues measure Then,

- *If* $\Lambda > 1$, *then the walk has positive speed almost surely,*
- *If* $\Lambda < 1$ *then the walk has zero speed almost surely.*

Moreover, in the second case,

$$\lim_{n\to\infty} \frac{\log|X_n|}{\log n} = \Lambda, \ a.s.$$

We also mention on this subject another work by E. Aidékon [2], concerning large deviations estimate, , as well as some work by G. Ben Arous, A. Fribergh, N. Gantert and A. Hammond [3], on the case where the tree can have leaves, .

If $\inf_{t\in[0,1]} \psi(t) < 0$, the walk (X_n) is positive recurrent for almost all environment; in this case, Y. Hu and Z. Shi [42] showed, under the additional assumptions that N is deterministic and that the law of A_i does not depend on i that $\frac{1}{\log n}\max_{0\le k\le n}|X_k|$ converges almost surely to an explicit constant.

We now turn to the critical case $\inf_{t\in[0,1]}\psi(t)=0$. We assume that there exists some $\delta>0$ such that

$$\psi(t)<\infty, \quad \forall\, t\in(-\delta,1+\delta), \qquad E_q(N(e)^{1+\delta})<\infty. \tag{2.5.1}$$

There are two different regimes in this case, depending on the sign of $\psi'(1):=\mathrm{e}^{-\psi(1)}E_q\{\sum_{i=1}^N A(e_i)\log A(e_i)\}$.

We begin with the case $\psi'(1)\ge 0$ (and $\inf_{t\in[0,1]}\psi(t)=0$). The walk is then extremely slow. Indeed Y. Hu and Z. Shi [44] proved, under the additional conditions that N is deterministic and that the law of A_i does not depend on i, that there exist constants $0<c_1\le c_2<\infty$ such that

$$c_1\le \liminf_{n\to\infty}\frac{\max_{0\le k\le n}|X_k|}{(\log n)^3}\le \limsup_{n\to\infty}\frac{\max_{0\le k\le n}|X_k|}{(\log n)^3}\le c_2, \qquad \text{a.s.} \tag{2.5.2}$$

We will actually prove that almost sure convergence holds in (2.5.2) if $\psi'(1)\ge 0$ (and $\inf_{t\in[0,1]}\psi(t)=0$). The limiting constant in (2.5.2), however, will have different natures depending on whether $\psi'(1)=0$ or $\psi'(1)>0$. We first state our result for the case $\psi'(1)=0$; in this case, condition $\inf_{t\in[0,1]}\psi(t)=0$ is equivalent to $\psi(1)=0$.

Theorem 2.5.4 *Assume $\psi(1)=\psi'(1)=0$. On the set of non-extinction,*

$$\lim_{n\to\infty}\frac{\max_{0\le k\le n}|X_k|}{(\log n)^3}=\frac{8}{3\pi^2\sigma^2}\,, \qquad \mathbb{P}_{\mathtt{MT}}\text{-a.s.},$$

where

$$\sigma^2:=E_q\Big\{\sum_{i=1}^N A(e_i)(\log A(e_i)^2\Big\}. \tag{2.5.3}$$

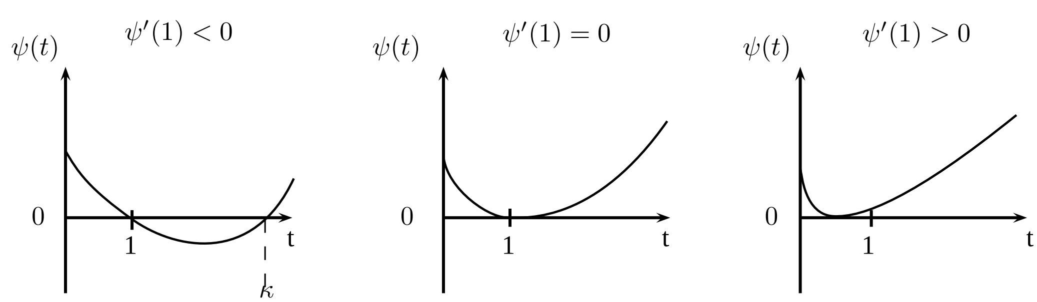


Figure 2.2: The different possible shapes for ψ

This result, as well as the following one, are part of a joint work with Y. Hu and Z. Shi.

Let us now turn to the case $\psi'(1) > 0$ (and $\inf_{t\in[0,\,1]} \psi(t) = 0$). In this case, there exists $0 < \theta < 1$ such that

$$\psi'(\theta) = 0. \tag{2.5.4}$$

[Thus, the first case corresponds to $\theta = 1$.]

Theorem 2.5.5 *Assume* $\inf_{t\in[0,\,1]} \psi(t) = 0$ *and* $\psi'(1) > 0$*. On the set of non-extinction,*

$$\lim_{n\to\infty} \frac{\max_{0\le k\le n} |X_k|}{(\log n)^3} = \frac{2\,\theta}{3\pi^2 \psi''(\theta)}\,, \qquad \mathbb{P}_{\mathtt{MT}}\text{-a.s.},$$

where $\theta \in (0,\, 1)$ *is as in (2.5.4) and* $\psi''(\theta) = E_q\Big\{ \sum_{i=1}^N A(e_i)^\theta\, (\log A(e_i))^2 \Big\}$.

Theorems 2.5.4 and 2.5.5 are proved in Chapter 4

When $\psi'(1) < 0$ (and $\inf_{t\in[0,\,1]} \psi(t) = 0$), then by defining $\kappa := \inf\{t > 1 :\, \psi(t) = 0\} \in (1,\, \infty]$ (with $\inf e := \infty$) and $\nu := 1 - \frac{1}{\min\{\kappa, 2\}} \in (0,\, \frac{1}{2}]$, the order of magnitude of $|X_n|$ is, loosely speaking, n^ν. Indeed Y. Hu and Z. Shi [44] showed (for identically distributed $A(e_i)$ and deterministic N), that

$$\lim_{n\to\infty} \frac{\log \max_{0<s<n} |X_s|}{\log n} = \nu,\ \mathbb{P}_{\mathtt{MT}} - a.s..$$

Note that when $\kappa \geq 2$, the walk behaves asymptotically like $n^{1/2}$. We precise this fact by a central limit theorem. Unfortunately we are not able to cover the whole regime $\kappa \in [2, \infty)$.

We first introduce the technical assumptions

$$\exists\ 0 < \varepsilon_0 < \infty;\, \forall i,\, \varepsilon_0 \le A(e_i) \le \frac{1}{\varepsilon_0},\ q - a.s. \tag{2.5.5}$$

and

$$\exists N_0\ ; N(e) \le N_0,\ q - p.s. \text{ and } P_q[N \ge 2 | A(e_1)] \ge \frac{1}{N_0} \tag{2.5.6}$$

Remark : The assumption (2.5.6) is in fact quite non-optimal; it suffices in fact to check (3.3.6) and the fact that the function g in (3.3.9) is bounded above and away from 0.

Note furthermore that those conditions imply (H1).

Theorem 2.5.6 *Suppose* $N(e) \geq 1$, $q - a.s.$*, (2.5.5), (2.5.6).*

If $p = 1$, $\rho'(1) < 0$ *and* $\kappa \in (8, \infty]$*, then there is a deterministic constant* $\sigma > 0$ *such that, for* MT *almost every tree* T*, the process* $\{|X_{\lfloor nt\rfloor}|/\sqrt{\sigma^2 n}\}$ *converges in law to the absolute value of a standard brownian motion, as* n *goes to infinity.*

Remark : This result is a generalization of a central limit theorem proved by Y. Peres and O. Zeitouni [78] in the case of a biased standard random walk on a Galton-Watson tree. In this case, $A(x)$ is a constant equal to $\frac{1}{m}$, therefore $\kappa = \infty$. Our proof is quite inspired from theirs.

In the annealed setting, things happen to be easier, and we can weaken the assumption on κ.

Theorem 2.5.7 *Suppose $N(e) \geq 1$, $q - a.s.$, (2.5.5), (2.5.6). If $p = 1$, $\rho'(1) < 0$ and $\kappa \in (5, \infty]$, then there is a deterministic constant $\sigma > 0$ such that, under $\mathbb{P}_{\mathtt{MT}}$, the process $\{|X_{\lfloor nt \rfloor}|/\sqrt{\sigma^2 n}\}$ converges in law to the absolute value of a standard brownian motion, as n goes to infinity.*

The last two theorems are proved in chapter 3

Remark : As we will see, the annealed CLT will even be true for $\kappa \in (2, \infty)$, on a different kind of trees, following a distribution that can be described as "the invariant distribution" for the Markov chain of the "environment seen from the particle".

We finally introduce a last model of random process in random environment, namely a continuous time extension.

2.6 The diffusion in a brownian potential.

Let $(W(x))_{x \in \mathbb{R}}$ be a one-dimensional brownian motion defined on $\mathbb{R}$ starting from 0, and, for $\kappa \in \mathbb{R}$,

$$W_\kappa(x) := W(x) - \frac{\kappa}{2} x.$$

Let $(\beta(t))_{t \geq 0}$ be another one-dimensional brownian motion, independent of W. We call *diffusion process with potential* W_κ a solution to the (formal) equation

$$dX_t = d\beta_t - \frac{1}{2} W'_\kappa(X_t) dt. \tag{2.6.1}$$

W'_κ has clearly no rigorous meaning, but a mathematical definition of (2.6.1) can be given in terms of the infinitesimal generator. For a given realization of W_κ, X_t is a real-valued diffusion started at 0 with generator

$$\frac{1}{2} e^{W_\kappa(x)} \frac{d}{dx} \left(e^{-W_\kappa(x)} \frac{d}{dx} \right).$$

This diffusion can also be defined by a time-change representation :

$$X_t = A_\kappa^{-1}\left(B(T_\kappa^{-1}(t))\right),$$

where

$$A_\kappa(x) = \int_0^x e^{W\kappa(y)}dy,$$

$$T_\kappa(t) = \int_0^t e^{-2W_\kappa(A_\kappa^{-1}(B(s)))}ds,$$

and B is a standard Brownian motion. A_κ is the scale function of this process, and its speed measure is $2e^{-W_\kappa(x)}dx$.

Intuitively, for a given environment W_κ, the diffusion X_t will tend to go to places where W_κ is low, and to spend a lot of time in the "valleys" of W_κ. If the environment is drifted ($\kappa > 0$), the process will be transient to the right, but it will be slowed by those valleys (see figure 2.6). This will be explained more precisely in section 5.2.

For general background on diffusion processes and time-change representation we refer to [83, 82, 47].

We will call $\mathcal{P}$ the probability associated to W, P_W the quenched probability associated to the diffusion, and $\mathbb{P} := \mathcal{P} \otimes P_W$ the annealed probability.

T. Brox gave a result concerning the long time behavior of the diffusion in the case $\kappa = 0$. Namely, under the probability $\mathbb{P}$,

$$\frac{X_t}{(\log t)^2} \overset{\text{(law)}}{\to} U,$$

where U follows an explicit distribution.

The case $\kappa > 0$ was studied both by K. Kawazu and H. Tanaka ([53]) and Y. Hu, Z. Shi, M. Yor ([46]) and exhibits a "Kesten-Kozlov-Spitzer" behavior: when $\kappa > 1$, the diffusion has a positive speed; when $\kappa = 1$, under $\mathbb{P}$,

$$\frac{X_t \log t}{t} \overset{(P)}{\to} 4,$$

while, when $0 < \kappa < 1$,

$$\frac{X_t}{t^\kappa} \overset{\text{(law)}}{\to} V$$

in distribution, where V follows the inverse of a completely asymmetric stable law.

We are interested in the deviations between X_t and its asymptotic behavior, in the case $0 < \kappa < 1$. Our motivation is the following : as seen above, this diffusion's asymptotic behavior is quite similar to the behavior of standard RWRE. We are going to show that

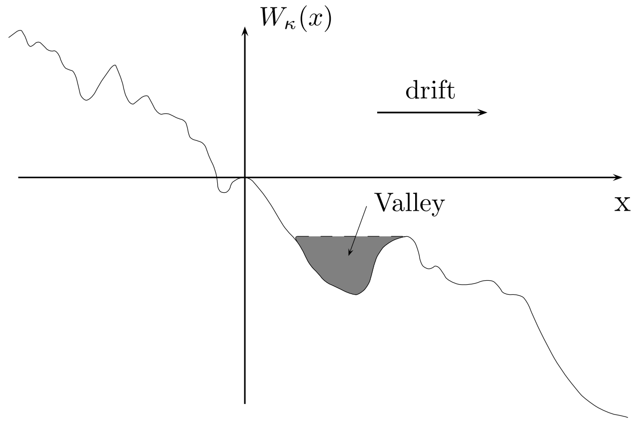


Figure 2.3: A “valley”.

this is also true concerning the deviations. The discrete case has been thoroughly studied by A. Fribergh, N. Gantert and S. Popov [37]. We have been quite inspired by this work, especially concerning the ”quenched slowdown” case. In the other cases the powerfull tools of stochastic calculus even allow us to get more precise estimates.

We mention that these questions have already been studied in the other cases, we refer to [41] for estimates in the case $\kappa = 0$, and to [94] for large deviation estimates in the case $\kappa > 1$.

Our study will split into four different problems, indeed the quenched and annealed settings present different behavior, and for each of them we have to consider deviations above the asymptotic behavior (or speedup) and deviations under the asymptotic behavior (or slowdown).

We start with the annealed results. For u and v two functions of t, we note $u \gg v$ if $u/v \to_{t\to\infty} \infty$.

Theorem 2.6.1 (Annealed speedup/slowdown) *Suppose $0 < \kappa < 1$, and $u \to \infty$ is a function of t such that for some $\varepsilon > 0$, $u \ll t^{1-\kappa-\varepsilon}$, then there exist two positive constants C_1 and C_2 such that*

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}\left(X_t > t^\kappa u\right)}{u^{\frac{1}{1-\kappa}}} = C_1, \tag{2.6.2}$$

and if $\log u \ll t^\kappa$,

$$\lim_{t\to\infty} u\mathbb{P}\left(X_t < \frac{t^\kappa}{u}\right) = C_2. \tag{2.6.3}$$

Furthermore, the results remain true if we replace X_t by $\sup_{s<t} X_s$.

This is in fact a easy consequence of the study of the hitting time of a certain level by the diffusion. We set $H(v) = \inf\{t > 0 : X_t = v\}$. We have the following estimates.

Theorem 2.6.2 *Suppose* $0 < \kappa < 1$ *and* $\varepsilon > 0$. *For* $u \to \infty$ $v \to \infty$ *two functions of* t *such that for some* $\varepsilon > 0$, $u \ll v^{1-\kappa-\varepsilon}$, *there exist two positive constants* C_1 *and* C_2 *such that*

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}\left[H(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right]}{u^{\frac{1}{1-\kappa}}} = C_1, \tag{2.6.4}$$

and if $\log u \ll v$,

$$\lim_{t\to\infty} u\mathbb{P}\left[H(v) > (vu)^{1/\kappa}\right] = C_2. \tag{2.6.5}$$

The proof of this result involves a representation of $H(v)$ introduced in [41].

We now turn to the quenched setting. We have the following estimates for the speedup

Theorem 2.6.3 (Quenched speedup) *Suppose* $0 < \kappa < 1$, *and* $u \to \infty$ *is a function of* t *such that for some* $\varepsilon > 0$, $u \ll t^{1-\kappa-\varepsilon}$, *then there exists a positive constants* C_3 *such that*

$$\lim_{t\to\infty} \frac{-\log P_W\left(X_t > t^\kappa u\right)}{u^{\frac{1}{1-\kappa}}} = C_3,\ P - a.s..$$

Furthermore the result remains true if we replace X_t *by* $\sup_{s\le t} X_s$.

As before, the proof of this will reduce to estimates on the hitting times.

Theorem 2.6.4 *For* $u \to \infty$ $v \to \infty$ *two functions of* t *such that for some* $\varepsilon > 0$, $u \ll v^{1-\kappa-\varepsilon}$, *then*

$$\lim_{t\to\infty} \frac{-\log P_W\left[H(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right]}{u^{\frac{1}{1-\kappa}}} = C_3,\ P - a.s.. \tag{2.6.6}$$

For the slowdown, our result is less precise.

Theorem 2.6.5 (Quenched slowdown) *Suppose* $\kappa > 0$. *Let* $\nu \in (0, 1 \wedge \kappa)$, *then*

$$\lim_{t\to\infty} \frac{\log(-\log P_W[H(t^\nu) > t])}{\log t} = \left(1 - \frac{\nu}{\kappa}\right) \wedge \frac{\kappa}{\kappa+1},\ P - a.s., \tag{2.6.7}$$

$$\lim_{t\to\infty} \frac{\log(-\log P_W[X_t < t^\nu])}{\log t} = \left(1 - \frac{\nu}{\kappa}\right) \wedge \frac{\kappa}{\kappa+1},\ P - a.s.. \tag{2.6.8}$$

These result are proved in chapter 5

2.7 Content

This thesis will be separated in three independent chapters.

In chapter 3 we first present the proof in the general case of the recurrence/transience criterion Theorem 2.5.1, as well as the proof of the criterion in the critical case 2.5.2. Then we present the proof of the Central Limit Theorems 2.5.6 and 2.5.7. The proofs of these results are rather similar, indeed both are divided in two parts, in the first part we introduce a new distribution of random trees, for which it will be easier to show a Central Limit Theorem, with a standard Brownian motion, instead of a reflected one, as limiting process. Then, in a second part we will use a coupling argument to deduce the theorem in the classical tree. These result can also be found in [33]

Chapter 4 contains the proof, obtained in collaboration with Y. Hu and Z. Shi, of Theorems 1.5.2 and 1.5.3. These results are based, in particular, on the study of the asymptotic of

$$\exp\Big(-\min_{|x|=n}\max_{y\in \,]\!]\varnothing, x]\!]} V(y)\Big). \tag{2.7.1}$$

We will see that this quantity is related to the height of the excursions of the walk in the tree. This work is also presented in [35].

Finally, in chapter 5 we show Theorems 2.6.1, 2.6.3 and 2.6.5. These results have been published in [34].

Chapter 3

The Central Limit theorem.

In this chapter we present the proof of Theorem 2.5.6. The proof will be separated into several steps

- In section 3.1 we prove Theorem 2.5.1 and Proposition 2.5.2.
- In section 3.2 we introduce a different kind of trees, on which we are able to compute an "invariant" distribution.
- In section 3.3 we show a central limit theorem for random walks on trees following the "invariant" distribution.
- In section 3.4 we expose a coupling between the original law and the new one.
- In section 3.5 we show some technical lemmas.
- In section 3.6 we show Theorem 2.5.7

3.1 Proof of Theorem 2.5.1.

Let us first introduce an associated martingale, which will be of frequent use in the sequence.

Let $\alpha \in \mathbb{R}^+$ and

$$Y_n^{(\alpha)} = \sum_{x \in T_n} \prod_{e < z \leq x} A(z)^\alpha = \sum_{x \in T_n} C_x^\alpha.$$

$Y_n^{(\alpha)}$ is known as Mandelbrot's Cascade.

It it is easy to see that if $\rho(\alpha) < \infty$ then $\frac{Y_n^{(\alpha)}}{\rho(\alpha)^n}$ is a non-negative martingale, with a.s. limit $Y^{(\alpha)}$.

We have the following theorem, due to J.D. Biggins (1977) (see [9, 10]) that allows us to know when $Y^{(\alpha)}$ is non trivial.

Statement 3.1.1 (Biggins) *Let $\alpha \in \mathbb{R}^+$. Suppose ρ is finite in a small neighborhood of α, and $\rho'(\alpha)$ exists and is finite, then the following are equivalent*

- *given non-extinction, $Y^{(\alpha)} > 0$ a.s.,*
- $P_{\mathtt{MT}}[Y^{(\alpha)} = 0] < 1$,
- $E_{\mathtt{MT}}[Y^{(\alpha)}] = 1$,
- $E_q\left[\left(\sum_0^{N(e)} A(e_i)^\alpha\right) \log^+ \left(\sum_0^{N(e)} A(e_i)^\alpha\right)\right] < \infty$, *and* *(H2)*$:= \alpha\rho'(\alpha)/\rho(\alpha) < \log \rho(\alpha)$,
- $\frac{Y^{(\alpha)}}{\rho(\alpha)}$ *converges in L^1.*

This martingale is related to some branching random walk, and has been intensively studied ([71, 9, 10, 61, 62, 73]). We will see that it is closely related to our problem.

Let us now prove Theorem 2.5.1. We shall use the following lemma, whose proof is similar to the proof presented in page 129 of [66] and omitted.

Lemma 3.1.1

$$\min_{0\le t\le 1} E\left[\sum_{x\in T_1} A(x)^t\right] = \max_{0<y\le 1} \inf_{t>0} y^{1-t} E\left[\sum_{x\in T_1} A(x)^t\right].$$

(1) Let us begin with the subcritical case, We suppose there exists some $0 < \alpha < 1$ such that $\rho(\alpha) = \inf_{0\le t<1} \rho(t) < 1$. Then, following [54] (Prop 9-131), and standard electrical/capacited network theory, if the conductances have finite sum, then the random walk is positive recurrent, the electrical network has zero conductance a.s., and the capacited network admits no flow a.s.. We have

$$\sum_{x\in T^*} C_x^\alpha = \sum_{n=0}^{\infty} \sum_{x\in T_n} C_x^\alpha = \sum_n \rho(\alpha)^n Y_n^{(\alpha)}.$$

Since $Y_n^{(\alpha)}$ is bounded (actually it converges to 0), we have

$$\sum_{x\in T^*} C_x^\alpha < \infty,\ \mathtt{MT} - a.s..$$

This implies that *a.s.*, for all but finitely many x, $C_x < 1$, and then $C_x \leq C_x^\alpha$, which gives the result.

(2) As before, we have α such that $\rho(\alpha) = \inf_{0\leq t\leq 1} \rho(t) \leq 1$. We have to distinguish two cases. Either $\rho'(1) \geq 0$, therefore it is easy to see that, for α, (H2) is not verified, so

$$\sum_{x\in T_n} C_x^\alpha = Y_n^{(\alpha)} \to 0,$$

when n goes to ∞ . Then for n large enough, $C_x < 1$ for every $x \in T_n$, whence

$$\sum_{x\in T_n} C_x \to 0,$$

then by the *max-flow min-cut* theorem, the associated capacited network admits no flow a.s., this implies that no electrical current flows, and that the random walk is recurrent `MT`-a.s..

We now deal with the case where $\rho'(1) < 0$, then $\alpha = 1$. The proof is similar to [66], but, as it is quite short, we give it for the sake of clarity. We use the fact that, if the capacited network admits no flow from e, then the walk is recurrent.

We call F the maximum flows from e in T, and for $x \in T$, $|x| = 1$, we call F_x the maximum flow in the subtree $T_x = \{y \in T, x \leq y\}$, with capacity $\frac{C_y}{A(x)}$ along the edge $(\overleftarrow{x}, x)$. It is easy to see that F and F_x have the same distribution, and that

$$F = \sum_{|x|=1} A(x)(F_x \wedge 1). \tag{3.1.1}$$

Taking the expectation yields

$$E[F] = E[F_x \wedge 1] = E[F \wedge 1],$$

therefore $ess \sup F \leq 1$. By independence, we obtain from (3.1.1) that

$$ess \sup F = (ess \sup \sum_{|x|=1} A(x))(ess \sup F).$$

This implies that $F = 0$ almost surely, as $(ess \sup \sum_{|x|=1} A(x)) > 1$, when $\sum_{|x|=1} A(x)$ is not identically equal to 1 .

(3) We shall use the fact that, if the water flows when C_x is reduced exponentially in $|x|$, then the electrical current flows, and the random walk is transient a.s. (see [64]).

We have

$$\inf_{\alpha\in[0,1]} E\left[\sum_{0}^{N(e)} A(e_i)^\alpha\right] = p > 1$$

(p can be infinite, in which case the proof still applies).

We introduce the measure μ_n defined as

$$\mu_n(A) = E[\sharp(A \cap \{\log C_x\}_{x \in T_n})],$$

where $\sharp$ denotes the cardinality.

One can easily check that

$$\phi_n(\lambda) := \int_{-\infty}^{+\infty} e^{\lambda t} d\mu_n(t) = E\left[\sum_{x \in T_n} C_x^\lambda\right] = \rho(\lambda)^n.$$

Let $y \in (0, 1]$ be such that $p = \inf_{t>0} y^{1-t} E[\sum_{x \in T_1} A(x)^t]$. Then, using Cramer-Chernov theorem (and the fact that the probability measure μ_n/m^n has the same Laplace transform as the sum of n independent random variables with law μ_1/m), we have

$$\frac{1}{n} \log \mu_n([n(-\log y), \infty)) \to \log(p/y).$$

Now, if we set $1/y < q < p/y$, there exists k such that

$$E[\sharp\{x \in T_k | C_x > y^k\}] > q^k.$$

Then the end of the proof is similar to the proof in [66]. We chose a small $\epsilon > 0$ such that,

$$E[\sharp\{x \in T_k | C_x > y^k, \text{ and } \forall e < z \leq x, A(z) > \epsilon\}] > q^k.$$

Let T^k be the tree whose vertices are $\{x \in T_{kn}, n \in \mathbb{N}\}$ such that $x = \overleftarrow{y}$ in T^k iff $x \leq y$ in T and $|y| = x + k$. We form a random subgraph $T^k(\omega)$ by deleting the edges (x, y) where

$$\prod_{x<z\leq y} A(z) < q^k \text{ or } \exists x < z \leq y, A(z) < \epsilon.$$

Let Γ_0 be the connected component of the root. The tree Γ_0 is a Galton-Watson tree, such that the expected number of children of a vertex is $q^k > 1$, hence with a positive probability Γ_0 is infinite and has branching number over q^k.

Using Kolmogoroff's 0-1 Law, conditionally to the survival there is almost surely a infinite connected component, not necessarily containing the root. This connected component has branching number at least q^k. Then we can construct almost surely a subtree T' of T, with branching number over q, such that $\forall x \in T'$, $A(x) > \epsilon$ and if $|x| = nk, |y| = (n + 1)k$ and $x < y$ then $\prod_{x<z\leq y} A(z) > q^k$. This implies the result.

We now turn to the proof of Proposition 2.5.2. Let π be an invariant measure for the Markov chain (X_n, P_T) (that is a measure on T such that, $\forall x \in T$, $\pi(x) = \sum_{y\in T} \pi(y)\omega(y,x)$), then one can easily check that

$$\pi(x) = \frac{\pi(e)\omega(e, \overleftarrow{e})}{\omega(x, \overleftarrow{x})} \prod_{0<z\leq x} A(z),$$

with the convention that a product over an empty set is equal to 1.

Then almost surely there exists a constant $c > 0$ (dependant of the tree) such that

$$\pi(x) > c\, C_x.$$

Thus

$$\sum_{x\in T} \pi(x) > c \sum_n Y_n^{(1)}.$$

-If $\rho'(1) < 0$, then (H2) is verified and $Y > 0$ a.s. conditionally to the survival of the system, thus the invariant measure is infinite and the walk is null recurrent.

-If $\rho'(1) = 0$, we use a recent result from Y. Hu and Z. Shi. In [45] it was shown that, under the assumptions of Theorem 2.5.2, there exists a sequence λ_n such that

$$0 < \liminf_{n\to\infty} \frac{\lambda_n}{n^{1/2}} \leq \limsup_{n\to\infty} \frac{\lambda_n}{n^{1/2}} < \infty$$

and $\lambda_n Y_n^{(1)} \to_{n\to\infty} Y$, with $Y > 0$ conditionally on the system's survival. The result follows easily.

-If $\rho'(1) > 0$, there exists $0 < \alpha < 1$ such that $\rho(\alpha) = 1$, $\rho'(\alpha) = 0$. We set, for every $x \in T$, $\tilde{A}(x) := A(x)^\alpha$. We set accordingly $\tilde{C}(x) = \prod_{0<z\leq x} \tilde{A}(z)$, and

$$\tilde{\rho}(t) := E_q\left[\sum_{i=1}^{N(e)} \tilde{A}(e_i)^t\right] = \rho(\alpha t).$$

Note that $\tilde{\rho}(1) = 1 = \inf_{0<t\leq 1} \rho(t)$ and $\tilde{\rho}'(1) = 0$. Note that under the ellipticity condition $\omega(x, \overleftarrow{x}) > \eta$, for some constant $c > 0$

$$\sum_{x\in T} \pi(x) < c \sum_{x\in T} C_x = \sum_{x\in T} \tilde{C}_x^{1/\alpha}.$$

Using Theorem 1.6 of [45] with $\beta = 1/\alpha$ and $\tilde{C}_x = e^{-V(x)}$, we get that for any $\frac{2}{3}\alpha < r < \alpha$,

$$E_{\mathtt{MT}}\left[\left(\sum_{x\in T_n} C_x\right)^r\right] = n^{-\frac{3r}{2\alpha}+o(1)}.$$

Note that as $r < 1$,

$$\left(\sum_n Y_n^{(1)}\right)^r \le \sum_n \left(Y_n^{(1)}\right)^r,$$

whence, using Fatou's Lemma,

$$E_{\mathtt{MT}}\left[\left(\sum_{x\in T} C_x\right)^r\right] < \infty.$$

This finishes the proof.

3.2 The IMT law.

We consider trees with a marked ray, which are composed of a semi infinite ray, called $Ray = \{v_0 = e, v_1 = \overleftarrow{v_0}, v_2 = \overleftarrow{v_1}...\}$ such that to each v_i is attached a tree. That way v_i has several children, one of which being v_{i-1}.

As we did for usual trees, we can "mark" these trees with $\{A(x)\}_{x\in T}$. Let $\tilde{\mathbb{T}}$ be the set of such trees.

Let $\mathcal{F}_n$ be the sigma algebra $\sigma(N_x, A_{x_i}, v_n \le x)$ and $\mathcal{F}_\infty = \sigma(\mathcal{F}_n, n \ge 0)$. While unspecified, "measurable" will mean "$\mathcal{F}_\infty$ - measurable".

Let $\hat{q}$ be the law on $\mathbb{N}\times\mathbb{R}_+^{*\mathbb{N}^*}$ defined by

$$\frac{d\hat{q}}{dq} = \sum_1^{N(e)} A(e_i).$$

Remark : For this definition to have any sense, it is fundamental that $E_q[\sum_1^{N(e)} A_i] = 1$, which is provided by the assumptions $\rho'(1) < 0$ and $p = 1$.

Following [78], let us introduce some laws on marked trees with a marked ray. Fix a vertex v_0 (the root) and a semi infinite ray, called *Ray* emanating from it. To each vertex $v \in Ray$ we attach independently a set of marked vertices with law $\hat{q}$, except to the root e to which we attach a set of children with law $(q+\hat{q})/2$. We chose one of these vertices, with probability $\frac{A(v_i)}{\sum A(v_i)}$, and identify it with the child of v on *Ray*. Then we attach a tree with law MT to the vertices not on *Ray*. We call IMT the law obtained.

We call $\theta^v T$ be the tree T "shifted" to v, that is, $\theta^v T$ has the same structure and labels as T, but its root is moved to vertex v.

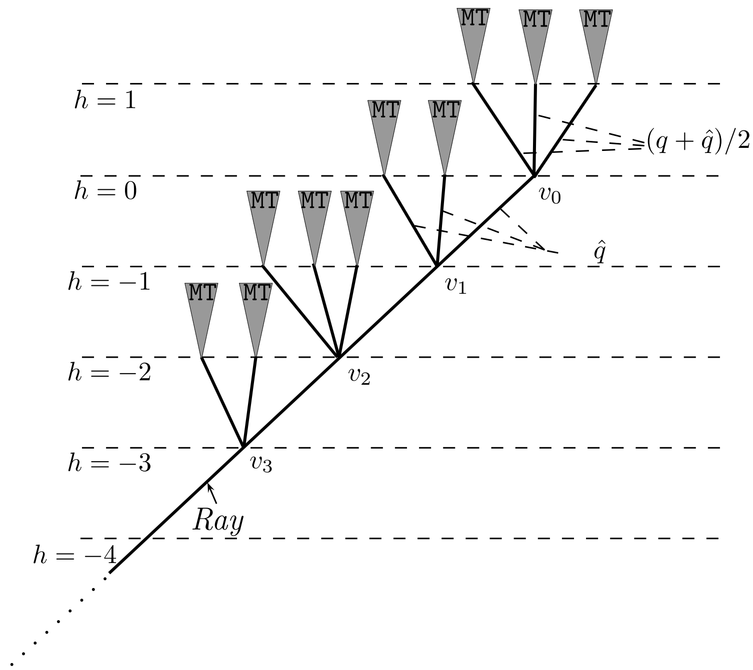


Figure 3.1: The IMT law.

Note that as before, given a tree T in $\tilde{\mathbb{T}}$, we can define in a unique way a family $\omega(x,y)$ such that $\omega(x,y) = 0$ unless $d(x,y) = 1$,

$$\forall x \in T, \sum_{y \in T} \omega(x,y) = 1,$$

and

$$\forall x \in T,\ A(x) = \frac{\omega(\overleftarrow{x}, x)}{\omega(\overleftarrow{x}, \overleftarrow{\overleftarrow{x}})}. \tag{3.2.1}$$

We call random walk on T the Markov chain $(X_t, \mathbb{P}_T)$ on T, starting from v_0 and with transition probabilities $(\omega(x,y))_{x,y \in T}$.

Let $T_t = \theta^{X_t} T$ denote the walk seen from the particle. T_t is clearly a Markov chain on $\tilde{\mathbb{T}}$. We set, for any probability measure μ on $\tilde{\mathbb{T}}$, $\mathbb{P}_\mu = \mu \otimes \mathbb{P}_T$ the annealed law of the random walk in a random environment on trees following the law μ. We have the following

Lemma 3.2.1 IMT *is a stationnary and reversible measure for the Markov process T_t, in the sense that, for every $F : \tilde{\mathbb{T}}^2 \to \mathbb{R}$ measurable,*

$$\mathbb{E}_{\texttt{IMT}}[F(T_0, T_1)] = \mathbb{E}_{\texttt{IMT}}[F(T_1, T_0)].$$

Proof : Suppose G is a $\mathcal{F}_n$-measurable function, that is, G only depends on the (classical) marked tree of the descendants of v_n, to which we will refer as T^{-n} and on the position of v_0 in the $n-th$ level of T^{-n}. We shall write accordingly $G(T)=G(T^{-n},v_0)$

We first show the following

Lemma 3.2.2 *If G is $\mathcal{F}_n$ measurable, then*

$$E_{\texttt{IMT}}[G(T)] = E_{\texttt{MT}}\left[\sum_{x\in T_n} C_x G(T,x)\left(\frac{1+\sum A(x_i)}{2}\right)\right]. \tag{3.2.2}$$

Remark : These formulae seem to create a dependency on n, which is actually irrelevant, since $E_q[\sum_{i=1}^{N(e)} A(e_i)]=1$.

Proof : This can be seen by an induction over n, using the fact that

$$E_{\texttt{IMT}}[G(T^{-n},v_0)] = E_q\left[\sum_{i=1}^{N} A(e_i) E\left[G(T'(i,N,A(e_j)),v_0)|i,N,A(e_j)\right]\right],$$

where $T'(x,N,A(e_i))$ is a tree composed of a vertex v_n with N children marked with the $A(e_i)$, and on each of this children is attached a tree with law `MT`, except on the i-th, where we attach a tree whose law is the same as $T^{-(n-1)}$.

Iterating this argument we have

$$E_{\texttt{IMT}}[G(T^{-n},v_0)] = E_{\texttt{MT}}\left[\sum_{x\in T_n} C_x E\left[G(T''(x,T),x)|x,T\right]\right],$$

where the n first levels of $T''(x,T)$ are similar to those of T, to each $y\in T''_n$, $x\neq y$ is attached a tree with law `MT`, and to x is attached a set of children with law $(\hat{q}+q)/2$, upon which we attach `MT` trees. The result follows.

Let us go back to the proof of Lemma 3.2.1. Using the definition of the random walk, we get

$$\mathbb{E}_{\texttt{IMT}}[F(T_0,T_1)] = E_{\texttt{IMT}}\left[\sum_{x\in T}\omega(v_0,x)F(T,\theta^x T)\right].$$

Suppose F is $\mathcal{F}_{(n-2)}\times\mathcal{F}_{(n-2)}$ measurable; then $T\to F(T,\theta^x T)$ is at least $F_{(n-1)}$ measurable. Then we can use (3.2.2) to get

$$\mathbb{E}_{\texttt{IMT}}[F(T_0,T_1)] = E_{\texttt{MT}}\left[\sum_{x\in T_n} C_x\left(\frac{1+\sum A(x_i)}{2}\right)\sum_{y\in T}\omega(x,y)F(T,\theta^y T)\right].$$

It is easily verified that

$$\forall x, y \in T, \omega(x,y)\frac{1+\sum A(x_i)}{2}C_x = \omega(y,x)\frac{1+\sum A(y_i)}{2}C_y.$$

Using this equality, we get

$$\begin{aligned}
\mathbb{E}_{\mathtt{IMT}}[F(T_0,T_1)] &= E_{\mathtt{MT}}\left[\sum_{x\in T_n}\sum_{y\in T}\omega(y,x)C_y\left(\frac{1+\sum A(y_i)}{2}\right)F((T,x),(T,y))\right] \\
&= E_{\mathtt{MT}}\left[\sum_{y\in T_{n+1}}\omega(y,\overleftarrow{y})C_y\left(\frac{1+\sum A(y_i)}{2}\right)F((T,\overleftarrow{y}),(T,y))\right] \\
&+ E_{\mathtt{MT}}\left[\sum_{y\in T_{n-1}}\sum_{i}\omega(y,y_i)C_y\left(\frac{1+\sum A(y_i)}{2}\right)F((T,y_i),(T,y))\right].
\end{aligned}$$

Using (3.2.2) and the fact that F is $\mathcal{F}_{(n-2)}\times\mathcal{F}_{(n-2)}$-measurable, we get

$$\begin{aligned}
\mathbb{E}_{\mathtt{IMT}}[F(T_0,T_1)] &= E_{\mathtt{IMT}}\left[\omega(e,\overleftarrow{e})F(\theta^{\overleftarrow{e}}T,T)\right] + E_{\mathtt{IMT}}\left[\sum_i \omega(e,e_i)F(\theta^{e_i}T,T)\right] \\
&= \mathbb{E}_{\mathtt{IMT}}\left[F(T_1,T_0)\right].
\end{aligned}$$

This finishes the proof of (3.2.1).

3.3 The Central Limit Theorem for the RWRE on IMT Trees.

In this section we introduce and show a central limit theorem for random walk on a tree following the law IMT. For $T \in \tilde{\mathbb{T}}$, let h be the horocycle distance on T (see Figure 2). h can be defined recursively by

$$\begin{cases} h(v_0) = 0 \\ h(\overleftarrow{x}) = h(x) - 1, \ \forall x \in T \end{cases}.$$

We have the following

Theorem 3.3.1 *Suppose $p = 1$, $\rho'(1) < 0$ and $\kappa \in [5,\infty]$, as well as assumptions (2.5.5) and (H2) or (2.5.6). There exists a deterministic constant σ such that, for $\mathtt{IMT}-$ a.e. T, the process $\{h(X_{\lfloor nt\rfloor})/\sqrt{\sigma^2 n}\}$ converges in distribution to a standard Brownian motion, as n goes to infinity.*

The proof of this result consists in the computation of a harmonic function S_x on T. We will show that the martingale S_{X_t} follows an invariance principle, and then that S_x stays very close to $h(x)$.

Let, for $v \in T$,

$$W_v = \lim_n \sum_{x\in T, v<x, d(v,x=n)} \prod_{v<z\leq x} A(z).$$

Statement 3.1.1 implies that $W_v > 0$ $a.s.$ and $E[W_v|\sigma(A(x_i), N(x), x < v)] = 1$. Now, let $M_0 = 0$ and if $X_t = v$,

$$M_{t+1} - M_t = \begin{cases} -W_v \text{ if } X_{t+1} = \overleftarrow{v} \\ W_{v_i}, \text{ if } X_{t+1} = v_i \end{cases}.$$

Given T, this is clearly a martingale with respect to the filtration associated to the walk. We introduce the function S_x defined as $S_e = 0$ and for all $x \in T$,

$$S_{x_i} = S_x + W_{x_i}, \tag{3.3.1}$$

in such a way that $M_t = S_{X_t}$.

Let

$$\eta = E_{\texttt{GW}}[W_0^2], \tag{3.3.2}$$

which is finite due to Theorem 2.1 of [61] (the assumption needed for this to be true is $\kappa > 2$). We call

$$V_t := \frac{1}{t}\sum_{i=1}^{t} \mathbb{E}_T[(M_{i+1} - Mi)^2|\mathcal{F}_t]$$

the normalized quadratic variation process associated to M_t. We get

$$\mathbb{E}_T[(M_{i+1} - Mi)^2|\mathcal{F}_t] = \omega(X_i, \overleftarrow{X_i})W_{X_i}^2 + \sum_{j=1}^{N(X_i)} \omega(X_i, X_{ij})W_{X_{ij}}^2 = G(T_i),$$

where X_{ij} are the children of X_i and G is a $L^1(\texttt{IMT})$ function on $\tilde{\mathbb{T}}$ (again due to $\kappa > 2$).

Let us define σ such that $E_{\texttt{IMT}}[G(T)] := \sigma^2\eta^2$. We have the following

Proposition 3.3.2 *The process $\{M\lfloor nt\rfloor/\sqrt{\sigma^2\eta^2 n}\}$ converges, for* IMT *almost every T, to a standard Brownian motion, as n goes to infinity.*

Proof : We need the fact that when t goes to infinity,

$$V_t \to \sigma^2\eta^2.$$

This comes from Birkhof's Theorem, using the transformation θ on $\tilde{\mathbb{T}}$, which conserves the measure IMT. The only point is to show that this transformation is ergodic, which follows from the fact that any invariant set must be independent of $\mathcal{F}_n^p = \sigma(N(x), A(x_i), v_n \leq x, h(x) < p)$, for all n, p, hence is independent of F_∞.

The result follows then from the Central Limit Theorem for martingales. Our aim is now to show that $h(X_t)$ and M_t/η stay close in some sense, then the central limit theorem for $h(X_t)$ will follow easily.

Let

$$\epsilon_0 < 1/100, \delta \in (1/2 + 1/3 + 4\epsilon_0, 1 - 4\epsilon_0)$$

and for every t, let ρ_t be an integer valued random variable uniformly chosen in $[t, t + \lfloor t^\delta \rfloor]$.

It is important to note that, by choosing ϵ_0 small enough, we can get δ as close to 1 as we need.

We are going to show the following

Proposition 3.3.3 *For any $0 < \epsilon < \epsilon_0$,*

$$\lim_{t\to\infty} \mathbb{P}_T(|M_{\rho_t}/\eta - h(X_{\rho_t})| \geq \epsilon\sqrt{t}) = 0, \ \mathtt{IMT} - a.s.,$$

further,

$$\lim_{t\to\infty} \mathbb{P}_T\left(\sup_{r,s<t,|r-s|<t^\delta} |h(X_r) - h(X_s)| > t^{1/2-\epsilon}\right) = 0, \ \mathtt{IMT} - a.s..$$

Before proving this result, we need some notations. For any vertex v of T, let

$$S_v^{\mathtt{Ray}} = \sum_{y \text{ on the geodesic connecting } v \text{ and } \mathtt{Ray}, y\notin\mathtt{Ray}} W_y.$$

We need a fundamental result on marked Galton-Watson trees. For a (classical) tree T, and x in T, set

$$S_x = \sum_{e<y\leq x} W_x,$$

with W_x as before, and

$$\mathbf{A}_n^\epsilon = \left\{v \in T, d(v,e) = n, \left|\frac{S_v}{n} - \eta\right| > \epsilon\right\}.$$

We have the following

Lemma 3.3.4 *Let $2 < \lambda < \kappa - 1$, then for some constant C_1 depending on ϵ,*

$$E_{\mathtt{MT}}\left[\sum_{x \in \mathbf{A}_n^\epsilon} C_x\right] < C_1 n^{1-\lambda/2}. \tag{3.3.3}$$

Proof : We consider the set $\mathbb{T}^*$ of trees with a marked path from the root, that is, an element of $\mathbb{T}^*$ is of the form $(T, v_0, v_1, ...)$, where T is in $\mathbb{T}$, $v_0 = e$ and $v_i = \overleftarrow{v_{i+1}}$.

We consider the filtration $F_k = \sigma(T, v_1, ...v_k)$. Given an integer n, we introduce the law $\widehat{\mathtt{MT}^*_\mathtt{n}}$ on $\mathbb{T}^*$ defined as follows : we consider a vertex e (the root), to this vertex we attach a set of marked children with law $\hat{q}$, and we chose one of those children as v_1, with probability $P(x = v_1) = A(x)/\sum A(e_i)$. To each child of e different from v_1 we attach independently a tree with law $\mathtt{MT}$, and on v_1 we iterate the process : we attach a set of children with law $\hat{q}$, we choose one of these children to be v_2, and so on, until getting to the level n. Then we attach a tree with law $\mathtt{MT}$ to v_n.

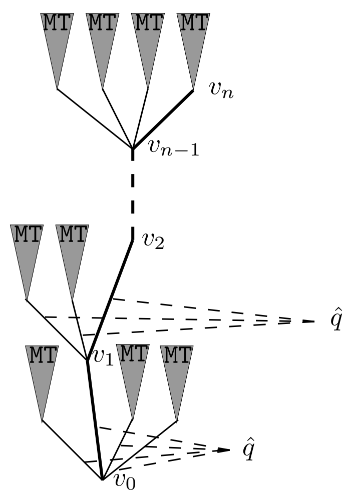


Figure 3.2: the law $\widehat{\mathtt{MT}^*_\mathtt{n}}$.

The same calculations as in the proof of Lemma 3.2.2 allow us to see the following fact : for f F_n-measurable,

$$E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[f(T, v_0, ..., v_n)] = E_{\mathtt{MT}}\left[\sum_{x \in T_n} C_x f(T, p(x))\right], \tag{3.3.4}$$

where $p(x)$ is the path from e to x. Note that, by construction, under $\widehat{\mathtt{MT}^*_\mathtt{n}}$ conditionally to $\tilde{F}^*_n := (C_{v_i}, 0 \le i \le n)$, the trees $T^{(v_i)}, 0 \le i \le n$ of the descendants of v_i who are not descendants of v_{i+1} are independent trees, and the law of $T^{(v_i)}$ is the law of a MT tree, except for the first level, whose law is $\hat{q}$ conditioned on v_{i+1}, $A(v_{i+1})$.

For a tree T in $\mathbb{T}^*$ we have

$$W_{v_k} = \sum_{v_k = \overleftarrow{x}, x \neq v_{k+1}} A(x) W_x + A(v_{k+1}) W_{v_{k+1}} := W_k^* + A(v_{k+1}) W_{v_{k+1}},$$

where

$$W_j* = \lim_{n\to\infty} \sum_{x\in T, v_j < x, v_{j+1} \not\leq x, d(v_j, x) = n} \prod_{v \leq z \leq x} A(z).$$

Iterating this, we obtain

$$W_{v_k} = \sum_{j=k}^{n-1} W_j^* \prod_{i=k+1}^{j} A(v_i) + W_{v_n} \prod_{i=k+1}^{n} A(v_i),$$

with the convention that the product over an empty space is equal to one. We shall use the notation $A_i := A(v_i)$ for a tree with a marked ray.

Finally, summing over k, we obtain

$$S_{v_n} = \sum_{j=0}^{n-1} W_j^* \sum_{k=0}^{j} \prod_{i=k+1}^{j} A_i + W_{v_n} \sum_{k=0}^{n} \prod_{i=k+1}^{n} A_i. \tag{3.3.5}$$

Let $B_j = \sum_{k=0}^{j} \prod_{i=k+1}^{j} A_i$. We note for simplicity $W_{v_n} := W_n^*$. Note that

$$E_{\widehat{\mathtt{MT}_n^*}}[W_0] = E_{\mathtt{MT}} \left[\left(\sum_{x \in T_n} C_x \right)^2 \right] := E_{\mathtt{MT}}[M_n^2]$$

converges to $\eta = E_{\mathtt{MT}}[W_0^2]$ as n goes to infinity. Indeed, recalling that $E_{\mathtt{MT}}[M_n] = 1$, we have

$$\begin{aligned} E_{\mathtt{MT}}[(M_{n+1}-1)^2] &= E_q \left[\left(\sum_{i=1}^{N(e)} A(e_i) U_i - 1 \right)^2 \right] \\ &= E_q \left[\left(\sum_{i=1}^{N(e)} A(e_i)(U_i - 1) + \sum_{i=1}^{N(e)} A(e_i) - 1 \right)^2 \right], \end{aligned}$$

where, conditionally to the A_i, U_i are i.i.d. random variables, with the same law as M_n. We get

$$E_{\mathtt{MT}}[(M_{n+1}-1)^2] = \rho(2) E_{\mathtt{MT}}[(M_n - 1)^2] + C_2,$$

where C_2 is a finite number. It is easy to see then that $E[M_n^2]$ is bounded, and martingale theory implies that M_n converges in L^2. Using the fact that $E_{\widehat{\mathtt{MT}^*_n}}[W_{v_k}] = E_{\widehat{\mathtt{MT}^*_{n-k}}}[W_0]$, a "Cesaro" argument implies that $E_{\widehat{\mathtt{MT}^*_n}}[S_{v_n}]/n$ converges to η as n goes to infinity. In view of that and (3.3.4) it is clear that, for n large enough

$$
\begin{aligned}
E_{\mathtt{MT}}\left[\sum_{x\in\mathbf{A}_n^\epsilon} C_x\right] &\leq E_{\mathtt{MT}}\left[\sum_{x\in T_n} C_x \mathbb{1}_{S_x - E_{\widehat{\mathtt{MT}^*_n}}[S_x] > n\epsilon/2}\right] \\
&\leq P_{\widehat{\mathtt{MT}^*_n}}\left[\left|S_{v_n} - E_{\widehat{\mathtt{MT}^*_n}}[S_{v_n}|\tilde{F}^*_n]\right| > \frac{n\epsilon}{4}\right] \\
&+ P_{\widehat{\mathtt{MT}^*_n}}\left[\left|E_{\widehat{\mathtt{MT}^*_n}}[S_{v_n}|\tilde{F}^*_n] - E_{\widehat{\mathtt{MT}^*_n}}[S_{v_n}]\right| > \frac{n\epsilon}{4}\right] := P_1 + P_2.
\end{aligned}
$$

Let us first bound P_1. Let $\tilde{W}^*_j := W^*_j - E_{\widehat{\mathtt{MT}^*_n}}[W^*_j|\tilde{F}^*_n]$ and $\lambda \in (2, \kappa - 1)$. We have

$$
\begin{aligned}
E_n^{(1)} &:= E_{\widehat{\mathtt{MT}^*_n}}\left[\left|S_{v_n} - E_{\widehat{\mathtt{MT}^*_n}}[S_{v_n}|\tilde{F}^*_n]\right|^\lambda\right] = E_{\widehat{\mathtt{MT}^*_n}}\left[\left|\sum_{i=0}^n \tilde{W}^*_i B_i\right|^\lambda\right] \\
&= E_{\widehat{\mathtt{MT}^*_n}}\left[E_{\widehat{\mathtt{MT}^*_n}}\left[\left|\sum_{i=0}^n \tilde{W}^*_i B_i\right|^\lambda |\tilde{F}^*_n\right]\right].
\end{aligned}
$$

Inequality from page 82 of [79] implies

$$
E_n^{(1)} \leq C(\lambda) n^{\lambda/2-1} E_{\widehat{\mathtt{MT}^*_n}}\left[\sum_{i=0}^n E_{\widehat{\mathtt{MT}^*_n}}\left[\left(\tilde{W}^*_i B_i\right)^\lambda |\tilde{F}^*_n\right]\right] \leq C_3 n^{\lambda/2-1} E_{\widehat{\mathtt{MT}^*_n}}\left[\sum_{i=0}^n B_i^\lambda\right],
$$

where we have admitted the following lemma

Lemma 3.3.5 $\forall\mu < \kappa$, *there exist some constant C such that*

$$
E_{\mathtt{MT}^*_n}[(W^*_i)^\mu|\tilde{F}^*_n] < C. \tag{3.3.6}
$$

moreover, there exists some $\varepsilon_1 > 0$ such that

$$
E_{\mathtt{MT}^*_n}[W^*_i|\tilde{F}^*_n] > \varepsilon_1.
$$

We postpone the proof of this lemma and finish the proof of Lemma 3.3.4. In order to bound $E_{\widehat{\mathtt{MT}^*}}\left[B_i^\lambda\right]$ we need to introduce a result from [10] (lemma 4.1).

Statement 3.3.1 (Biggins and Kyprianou) *For any $n \geq 1$ and any measurable function G,*

$$E_{\mathtt{MT}}\left[\sum_{x\in T_n} C_x G(C_y, e < y \leq x)\right] = E[G(e^{S_i}; 1 \leq i \leq n)],$$

where S_n is the sum of n i.i.d variables whose common distribution is determined by

$$E[g(S_1)] = E_q\left[\sum_{i=1}^{N(e)} A(e_i) g(\log A(e_i))\right]$$

for any positive measurable function g.

In particular, $E[e^{\lambda S_1}] = E_q[\sum_{i=1}^{N(e)} A(e_i)^{\lambda+1}] = \rho(\lambda+1) < 1$. We are now able to compute

$$E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}\left[B_n^\lambda\right] = E_{\mathtt{MT}}\left[\sum_{x\in T_n} C_x \left(\sum_{e\leq y\leq x}\prod_{y<z\leq x} A(z)\right)^\lambda\right] = E\left[\left(\sum_{k=0}^{n} e^{S_n - S_k}\right)^\lambda\right].$$

Using Minkowski's Inequality, we get

$$E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}\left[B_n^\lambda\right] \leq \left(\sum_{k=0}^{n} E\left[e^{\lambda(S_k - S_n)}\right]^{\frac{1}{\lambda}}\right)^\lambda \leq \left(\sum_{k=0}^{n} \rho(\lambda+1)^{\frac{n-k}{\lambda}}\right)^\lambda \leq C_4. \tag{3.3.7}$$

We can now conclude,

$$E_n^{(1)} \leq C_5 n^{\lambda/2},$$

and by Markov's Inequality,

$$P_1 < C_6/(\epsilon^\lambda n^{\lambda/2}). \tag{3.3.8}$$

Now we are going to deal with

$$P_2 = P_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}\left[\left|E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[S_{v_n}|\tilde{F}^*_n] - E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[S_{v_n}]\right| > n\epsilon/2\right].$$

Lemma 3.3.5 implies that $E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[W^*_j|\tilde{F}^*_n]$ is bounded above and away from zero, and a deterministic function of A_{j+1}. We shall note accordingly

$$E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[W^*_j|\tilde{F}^*_n] := g(A_{j+1}). \tag{3.3.9}$$

Recalling (3.3.5), we have

$$E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[S_{v_n}|\tilde{F}^*_n] = \sum_{j=0}^{n} E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[W^*_j|\tilde{F}^*_n])B_j = \sum_{0\leq j\leq k\leq n}\prod_{i=j}^{k} A_i g(A_{k+1}).$$

with the convention $g(A_{n+1}) = 1$ and $A_0 = 1$. We set accordingly

$$E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[S_{v_n}|\tilde{F}^*_n] := F(A_1, ..., A_n).$$

Recalling that, due to Statement 3.3.1, under the law $\widehat{\mathtt{MT}^*_\mathtt{n}}$, the A_i are i.i.d random variables we get

$$E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[F(A_1, ..., A_n)] = \sum_{0\le j\le k\le n} \prod_{i=j}^{k} E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[A_i] E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[g(A_{k+1})].$$

For $m \geq 0$ we call

$$\begin{aligned} &F^m[A_{m+1}, ..., A_n] \\ &\quad := \sum_{\substack{0\le j\le k\le n \\ k\le m-1}} \prod_{i=j}^{k} E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[A_i] E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[g(A_{k+1})] + \sum_{\substack{0\le j\le k\le n \\ k\ge m}} \prod_{i=j}^{m} E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[A_i] \prod_{i'=m+1}^{k} A_{i'} g(A_{k+1}). \end{aligned}$$

Note that $F^0 = F$ and $F^n = E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[S_{v_n}]$, thus we can write

$$\begin{aligned} E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[S_{v_n}|\tilde{F}^*_n] - E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[S_{v_n}] &= F^0(A_1, ...A_n) - F^n \\ &= F^0(A_1, ...A_n) - F^1(A_2, ...A_n) \\ &+ F^1(A_2, ...A_n) - F^2(A_3, ...A_n).... \\ &+ F^{n-1}(A_n) - F^n. \end{aligned}$$

We introduce the notations $\rho := E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[A_1] = \rho(2) < 1$, and for a random variable X, $\tilde{X} := X - E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[X]$.

The last expression gives us

$$\begin{aligned} &E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[S_{v_n}|\tilde{F}^*_n] - E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[S_{v_n}] \\ &\quad = \tilde{g}(A_1) + \tilde{A}_1(g(A_2) + A_2 g(A_3) + ... + \prod_{i=2}^{n} A_i g(A_{n+1})) \\ &\quad + \rho\tilde{g}(A_2) + \tilde{A}_2(1+\rho)\left[\sum_{j=3}^{n}\prod_{i=3}^{j} A_i g(A_{j+1})\right] \\ &\quad + \rho^2\tilde{g}(A_3) + \tilde{A}_3(1+\rho+\rho^2)\left[\sum_{j=4}^{n}\prod_{i=4}^{j} A_i g(A_{j+1})\right] + ... \\ &\quad + \rho^{n-1}\tilde{g}(A_n) + \tilde{A}_n(1+\rho+\rho^2+...\rho^{n-1}). \end{aligned}$$

We deduce easily that

$$\left|E_{\widehat{\mathtt{MT}^*_n}}[S_{v_n}|\tilde{F}^*_n] - E_{\widehat{\mathtt{MT}^*_n}}[S_{v_n}]\right| < C_7 + C_8\left|\sum_{k=1}^{n}\tilde{A}_k D_k(1+\rho+\rho^2+...\rho^{k-1})\right|, \tag{3.3.10}$$

where C_7, C_8 are finite constants and

$$D_k = \sum_{j=k+1}^{n}\prod_{i=k+1}^{j} A_i g(A_{j+1}).$$

To finish the proof of Lemma 3.3.4, we need to show that for every $\epsilon > 0$, $P_{\widehat{\mathtt{MT}^*_n}}[\sum_{k=1}^n \tilde{A}_k D_k(1+\rho+\rho^2+...\rho^{k-1}) > n\epsilon] < \frac{C(\epsilon)}{n^{\lambda/2-1}}$.

Recalling that $\lambda < \kappa - 1$, we can find a small $\nu > 0$ such that $\lambda(1+\nu) < \kappa - 1$. Then we have, by Minkowski's Inequality

$$\begin{aligned} E_{\widehat{\mathtt{MT}^*_n}}\left[D_k^{\lambda(1+\nu)}\right] &\le \left(\sum_{j=k+1}^{n}\left(E_{\widehat{\mathtt{MT}^*_n}}\left[C_8\prod_{i=k+1}^{n}A_i^{\lambda(1+\mu)}\right]\right)^{1/\lambda(1+\nu)}\right)^{\lambda(1+\nu)} \\ &\le \left(\sum_{j=k}^{n}\left(C_9\rho(1+\lambda(1+\mu))^{n-k+1}\right)^{1/\lambda(1+\nu)}\right)^{\lambda(1+\nu)} < C_{10}. \end{aligned} \tag{3.3.11}$$

Markov's Inequality then implies

$$P_{\widehat{\mathtt{MT}^*_n}}\left[\max_{k\le n} D_n > (\epsilon^2 n)^{\frac{1}{2(1+\nu)}}\right] \le C_{11}\frac{n}{n^{\lambda/2}\epsilon^{\lambda}}. \tag{3.3.12}$$

On the other hand, we call for $0 \le k \le n$,

$$N_k := \sum_{j=n-k}^{n} D_j g(A_{n+1})(1+\rho+\rho^2+...\rho^{j-1}).$$

It is easy to check that N_k is a martingale with respect to the filtration $\mathcal{H}_k = \sigma(A_j, n-k \le j \le n)$. We can compute the quadratic variation of this martingale

$$\langle N_k\rangle := \sum_{j=1}^{k} E_{\widehat{\mathtt{MT}^*_n}}[(N_k - N_{k-1})^2|\mathcal{H}_{k-1}] = \rho(3)\sum_{j=1}^{k}(D_{n-j})^2.$$

On the other hand, the total quadratic variation of N_k is equal to

$$[N_k] := \sum_{j=1}^{k}(N_k - N_{k-1})^2 = \sum_{j=1}^{k}(\tilde{A}_{n-j}D_{n-j})^2.$$

It is easy to check that if the event in (3.3.12) is fullfilled, then there exists some constant C_{12} such that $\langle N_k \rangle < C_{12} n^{1+\frac{1}{2(1+\nu)}}$ and $[N_k] < C_{12} n^{1+\frac{1}{2(1+\nu)}}$. Therefore, using (3.3.12) and Theorem 2.1 of [5],

$$P_{\widehat{\mathtt{MT}^*_n}}[|\sum_{k=1}^{n} \tilde{A}_k D_k| > n\epsilon] \le C_{11} \frac{n}{n^{\lambda/2}\epsilon^{\lambda}} + 2\exp - \frac{(\epsilon n)^2}{2C_{12} n^{1+\frac{1}{2(1+\nu)}}}. \tag{3.3.13}$$

Putting together (3.3.8) and (3.3.13), we obtain (3.3.3). This finishes the proof of Lemma 3.3.4. In particular, if $\kappa > 5$, we can choose $\lambda > 4$, so that

$$E_{\mathtt{MT}}[\sum_{x \in \mathbf{A}_n^{\epsilon}} C_x] < n^{-\mu},$$

with $\mu > 1$. The following corollary is a direct consequence of the proof.

Corollary 3.3.6 *For every $a > 0$ and $2 < \lambda < \kappa - 1$,*

$$P_{\widehat{\mathtt{MT}^*_n}}[|S_{v_k} - k\eta| > a] \le C_1 \frac{k^{1-\lambda/2}}{a^{\lambda}}.$$

We now give the proof of Lemma 3.3.5. As we said in the introduction, for this lemma we need either the assumption (H2) or the assumption (2.5.6). We give the proof in both cases. Note that, by construction of $\mathtt{MT}^*$, as, using Theorem 2.1 of [61], for every x a child of v_i, different from v_{i+1}, $W(x)$ has finite moments of order μ,

$$\begin{aligned} E_{\mathtt{MT}^*_n}[(W_i^*)^{\mu} | \tilde{F}_n^*] &= C_0 E_{\mathtt{MT}^*_n}\left[\left(\sum_{\overleftarrow{x} = v_i, x \neq v_{i+1}} A(x) \right)^{\mu} | \tilde{F}_n^* \right] & (3.3.14) \\ &= C_0 E_{\hat{q}} \left[\left(\sum_{|x|=1, x \neq v_1} A(x) \right)^{\mu} |A(v_1) \right] & (3.3.15) \end{aligned}$$

Note that the upper bound is trivial under assumption (2.5.6). We suppose (H2), Let f be a measurable test function, we have by construction

$$\begin{aligned} & E_{\hat{q}} \left[\left(\sum_{|x|=1, x \neq v_1} A(x) \right)^{\mu} f(A(v_1)) \right] \\ = \;& E_q \left[\sum_{i=1}^{N(e)} A(e_i) \left(\sum_{i \neq j} A(e_j) \right)^{\mu} f(A(e_i)) \right] \\ \le \;& \sum_{n=1}^{\infty} P_q(N(e) = n) E_q \left[\sum_{i=1}^{n} A'(i) \left(\sum_{i \neq j} A'(1) \right)^{\mu} f(A'(1)) \right] \end{aligned}$$

By standard convexity property, we get that the last term is lesser or equal to

$$\sum_{n=1}^{\infty} P_q(N(e)=n)E_q\left[\sum_{i=1}^{n} A'(i)n^{\mu-1}\sum_{i\neq j} A'(j)^{\mu} f(A'(i))\right]$$
$$\leq E_q[A'(1)^{\mu}]\sum_{n=1}^{\infty} P_q(N(e)=n)n^{\mu+1}E_q\left[A'(i)f(A'(i))\right]$$
$$= E_q[A'(1)^{\mu}]E_q\left[A'(i)f(A'(i))\right]E_q\left[N(e)^{\mu+1}\right],$$

while, still by construction

$$\begin{aligned} E_{\hat{q}}\left[f(A(v_1))\right] &= E_q\left[\sum_{i=1}^{N(e)} A(e_i)f(A(e_i))\right] \\ &= \sum_{n=1}^{\infty} P_q(N(e)=n)E_q[\sum_{i=1}^{n} A'(i)f(A'(i))] \\ &= E_q[N(e)]E_q[A'(1)f(A'(1))]. \end{aligned}$$

Therefore the result is direct. To prove the lower bound we begin with assumption (2.5.6). Actually we will only use the second part of this assumption, which is trivially implied by (H2), so the proof will also work for this case.

We have

$$E_{\hat{q}}\left[\sum_{|x|=1,x\neq v_1} A(x)f(A(v_1))\right] = E_q\left[\sum_{i=1}^{N(e)} A(e_i)\left(\sum_{i\neq j} A(e_j)\right) f(A(e_i))\right]$$
$$\begin{aligned} &\geq \epsilon_0 \sum_{i=1}^{\infty} E_q[A(e_i)f(A(e_i))\mathbb{1}_{\{i\leq N(e)\}}(N(e)-1)] \\ &\geq \epsilon_0 \sum_{i=2}^{\infty} E_q[A(e_i)f(A(e_i))\mathbb{1}_{\{i\leq N(e)\}}(N(e)-1)] + \epsilon_0 E_q[A(e_1)f(A(e_1))(N(e)-1)] \\ &\geq \epsilon_0 \sum_{i=2}^{\infty} E_q[A(e_i)f(A(e_i))\mathbb{1}_{\{i\leq N(e)\}}] + \epsilon_0 E_q[A(e_1)f(A(e_1))P(N(e)>2|A(e_1))] \\ &\geq \frac{\epsilon_0}{N_0}E_q\left[\sum_{i=1}^{N(e)} A(e_i)f(A(e_i)\right] = E_{\hat{q}}\left[f(A(v_1))\right], \end{aligned}$$

indeed for $i\geq 2$, the event $\{i<N(e)\}$ implies $N(e)-1>1$. This finishes the proof of Lemma 3.3.5.

Let us go back to IMT trees. We consider the following sets

$$\mathbf{B}_n^{\epsilon} = \left\{ v \in T, d(v, \mathtt{Ray}) = n, \left| \frac{S_v^{\mathtt{Ray}}}{n} - \eta \right| > \epsilon \right\}. \tag{3.3.16}$$

We can now prove the following

Lemma 3.3.7

$$\lim_{t\to\infty} \mathbb{P}_T(X_{\rho_t} \in \cup_{n=1}^{\infty} \mathbf{B}_n^{\epsilon}) = 0, \ \mathtt{IMT} - a.s..$$

Proof : we recall that a IMT tree is composed of a semi-infinite path from the root : $Ray = \{v_0 = e, v_1 = \overleftarrow{v_0} ...\}$, and that

$$W_j^* = \lim_n \sum_{x \in T, v_j < x, v_{j-1} \not\leq x, d(v_j, x = n)} \prod_{v \leq z \leq x} A(z).$$

Recalling Lemma 3.3.5, under IMT, conditionally to $\{Ray, A(v_i)\}$, W_j^* are independent random variables and $E[W_j^*] > \varepsilon_0$.

Let $1/2 < \gamma < \delta$. For a given tree T, we consider the event

$$\Gamma_t = \{\exists u \leq 2t | X_u = v_{\lfloor t^\gamma \rfloor}\}.$$

We have

$$\Gamma_t \subset \{\inf_{u \leq 2t} M_u \leq S_{v_{\lfloor t^\gamma \rfloor}}\},$$

and IMT almost surely, for some ϵ,

$$S_{v_{\lfloor t^\gamma \rfloor}} \leq - \sum_0^{\lfloor t^\gamma \rfloor} W_j^* < -\epsilon t^\gamma, \text{for t large enough.}$$

Since M_t is a martingale with bounded normalized quadratic variation V_t, we get that, for IMT almost every tree T,

$$\mathbb{P}_T(\Gamma_t) \to 0.$$

Going back to our initial problem, we have

$$\begin{aligned} \mathbb{P}_T(X_{\rho_t} \in \cup_{m=1}^{\infty} \mathbf{B}_m^{\epsilon}) &\leq \mathbb{P}_T(X_{\rho_t} \in \cup_{n=1}^{\infty} \mathbf{B}_m^{\epsilon}; \Gamma_t^c) + \mathbb{P}_T(\Gamma_t) & (3.3.17) \\ &\leq \frac{1}{\lfloor t^\delta \rfloor} \mathbb{E}_T \left[\sum_{s=0}^{H_{v_{\lfloor t^\gamma \rfloor}}} \mathbb{1}_{X_s \in \cup_{m=1}^{\infty} \mathbf{B}_m^{\epsilon}} \right] + \mathbb{P}_T(\Gamma_t), & (3.3.18) \end{aligned}$$

where $H_{v_{\lfloor t^\gamma \rfloor}}$ is the first time the walk hits $v_{\lfloor t^\gamma \rfloor}$.

As before we call $T^{(v_i)}$ the subtree constituted of the vertices $x \in T$ such that $v_i \leq x \not\leq x$. The first part of the right hand term of (3.3.17) is equal to

$$\frac{1}{\lfloor t^\delta \rfloor} \mathbb{E}_T \left[\sum_{i=0}^{\lfloor t^\gamma \rfloor} \sum_{s=0}^{H_{v_{\lfloor t^\gamma \rfloor}}} \mathbb{1}_{X_s \in \cup_{m=1}^\infty \mathbf{B}_m^\epsilon \cap T^{(v_i)}} \right] \leq \frac{1}{\lfloor t^\delta \rfloor} \sum_{i=0}^{\lfloor t^\gamma \rfloor} \mathbb{E}_T \left[\sum_{s=0}^{H_{v_{\lfloor t^\gamma \rfloor}}} \mathbb{1}_{X_s = v_i} \right] N_i,$$

where N_i is the P_T-expectation of the number of visits to $\cup_{n=1}^\infty \mathbf{B}_m^\epsilon \cap T^{(v_i)}$ during one excursion in $T^{(v_i)}$. Lemma 3.3.4 implies that, under IMT conditioned on $\{Ray, A(v_i)\}$, N_i are independent and identically distributed variables, with finite expectation, up to a bounded constant due to the first level of those subtrees. We are now going to compute $\mathbb{E}_T \left[\sum_{s=0}^{H_{v_{\lfloor t^\gamma \rfloor}}} \mathbb{1}_{X_s = v_i} \right]$. Given T, we have

$$\sum_{s=0}^{H_{v_{\lfloor t^\gamma \rfloor}}} \mathbb{1}_{X_s = v_i} \leq 1 + M_i,$$

where M_i is the number of times the walk, leaving from v_i, gets back to v_i before hitting $v_{\lfloor t^\gamma \rfloor}$. M_i follows a geometric law, with parameter $p_i = \mathbb{P}_T^{v_i}[H_{v_{\lfloor t^\gamma \rfloor}} < H_{v_i}]$.

Standard computations for random walks on $\mathbb{Z}$, (see, for example, Theorem 2.1.12 of [99]) imply that

$$p_i = \frac{\omega(v_i, v_{i+1})}{1 + \sum_{j=i}^{\lfloor t^\gamma \rfloor - 1} \prod_{k=j-1}^{\lfloor t^\gamma \rfloor} A(v_k)},$$

and, going back to our initial problem,

$$\begin{aligned} \mathbb{P}_T(X_{\rho_t} \in \cup_{m=1}^\infty \mathbf{B}_m^\epsilon) &\leq \mathbb{P}_T(\Gamma_t) + \frac{C_{14}}{\lfloor t^\delta \rfloor} \sum_{i=0}^{\lfloor t^\gamma \rfloor} \left(1 + \sum_{j=i}^{\lfloor t^\gamma \rfloor - 1} \prod_{k=j-1}^{\lfloor t^\gamma \rfloor} A(v_k) \right) N_i \\ &\leq \mathbb{P}_T(\Gamma_t) + V_t \frac{C_{14}}{\lfloor t^\delta \rfloor} \sum_{i=0}^{\lfloor t^\gamma \rfloor} N_i, \end{aligned}$$

with $V_t = 1 + \sum_{j=0}^{\lfloor t^\gamma \rfloor - 1} \prod_{k=j-1}^{\lfloor t^\gamma \rfloor} A(v_k)$.

As in the proof of Lemma 3.3.4, statement 3.3.1 implies that $E_{\mathtt{IMT}}[V_t^\alpha] < C_{15}$ for some $\alpha > 2$. Now we can choose δ close to one and γ close to $1/2$, and μ such that $1/\alpha < \mu < \delta - \gamma$

Markov's Inequality and the Borel Cantelli Lemma imply that, IMT-almost surely, there exists t_0 such that $\forall t > t_0, V_t \leq t^\mu$, and then,

$$\mathbb{P}_T(X_{\rho_t} \in \cup_{n=1}^\infty \mathbf{B}_m^\epsilon) \leq \mathbb{P}_T(\Gamma_t) + \frac{C_{16}}{\lfloor t^{\delta - \mu} \rfloor} \sum_{i=0}^{\lfloor t^\gamma \rfloor} N_i.$$

Since $\delta - \mu < \gamma$, an application of the law of large numbers finishes the proof of Lemma 3.3.7.

We are now able to prove the first part of Proposition 3.3.3. Note that under $\mathtt{IMT}$, S_{v_n} follows the same law as S_{v_n} in a $\mathbb{T}^*$ tree under $\widehat{\mathtt{MT}_n^*}$, whence

$$S_{v_n}/n \underset{n\to\infty}{\to} -\eta$$

in probability. Let Q_t be the first ancestor of X_{ρ_t} on *Ray*. Statement 3.3.1 and standard RWRE theory imply that Q_t is transient, therefore

$$S_{Q_t}/h(Q_t) \underset{t\to\infty}{\to} \eta,$$

so that, for any positive ϵ_1, for large t,

$$|S_{Q_t}/\eta - h(Q_t)| \leq \epsilon_1 \sup_{s\leq 2t} |M_t|. \tag{3.3.19}$$

We can now compute

$$|M_{\rho_t}/\eta - h(X_{\rho_t})| = |S^{\mathtt{Ray}}_{X_{\rho_t}}/\eta - d(X_{\rho_t}, \mathtt{Ray}) + S_{Q_t}/\eta - h(Q_t)|.$$

In view of (3.3.19) on the event $\{X_{\rho_t} \notin \cup_{n=1}^{\infty} \mathbf{B}^{\epsilon}_m\}$, we have

$$|M_{\rho_t}/\eta - h(X_{\rho_t})| \leq 2\epsilon_1 \sup_{s\leq 2t} |M_s|.$$

The process V_t being bounded $\mathtt{IMT}$ *a.s.*, a standard martingale inequality implies

$$\lim_{\epsilon_1\to 0} \limsup_{t\to\infty} \mathbb{P}^0_T(\sup_{s\leq t} |M_s| > \epsilon\sqrt{t}/(2\epsilon_1)) = 0.$$

It follows that

$$\lim_{t\to\infty} \mathbb{P}_T(|M_{\rho_t}/\eta - h(X_{\rho_t})| \geq \epsilon\sqrt{t}) = 0, \mathtt{IMT} - a.s.$$

We are now going to prove the second part of Proposition 3.3.3. The course of the proof is similar to [78]. We have the following lemma

Lemma 3.3.8 *for any u, $t \geq 1$,*

$$\mathbb{P}_{\mathtt{MT}}(|X_i| \geq u \text{ for some } i \leq t) \leq 2te^{-u^2/2t}.$$

Proof : We consider the graph T^* obtained by truncating the tree T after the level $u-1$, and adding an extra vertex e^*, connected to all vertices in T_{u-1}. We construct a random walk X^*_s on T^* as following

$$\mathbb{P}^0_T(X^*_{i+1}=y|X^*_i=x)=\begin{cases}\omega(x,y) \text{ if } |x|<u-1 \text{ or } |x|=u-1, |y|=u-2\\ 1-\omega(x,\overleftarrow{x}) \text{ if } |x|=u-1, y=e^*\\ \tilde{\omega}(e^*,y) \text{ if } x=e^*, |y|=u-1\end{cases}.$$

We can choose $\tilde{\omega}(e^*,y)$ arbitrarily, provided $\sum_{y\in T_{u-1}}\tilde{\omega}(e^*,y)=1$, so we will use this choice to ensure the existence of an invariant measure : indeed, if π is an invariant measure for the walk, one can easily check that, for any x such that $|x|\le u-1$, calling $x^{(1)}$ the first vertex on the path from e to x,

$$\pi(x)=\frac{\pi(e)\omega(e,x^{(1)})}{\omega(x,\overleftarrow{x})}\prod_{x^{(1)}<z\le x}A(z).$$

Further, we need that, for every $x\in T_{u-1}$,

$$\pi(x)(1-\omega(x,\overleftarrow{x}))=\pi(e^*)\tilde{\omega}(e^*,x).$$

Summing over x, and using $\sum_{y\in T_u}\tilde{\omega}(e^*,y)=1$, we get

$$\begin{aligned}\pi(e^*)&=\pi(e)\sum_{x\in T_{u-1}}\omega(e,x^{(1)})\prod_{x^{(1)}<z\le x}A(z)\frac{\sum\omega(x,x_i)}{\omega(x,\overleftarrow{x})}\\ &\le\pi(e)\sum_{x\in T_u}\prod_{x^{(1)}<z\le x}A(z)\le\pi(e)Y_u.\end{aligned}$$

Then,

$$\mathbb{P}_{\mathtt{MT}}(\exists i\le t,\, X_i\ge u)\le\mathbb{P}_{\mathtt{MT}}(\exists i\le t,\, X^*_i=e^*)\le\sum_{i=1}^t\mathbb{P}_{\mathtt{MT}}(X^*_i=e^*).$$

By the Carne-Varnopoulos Bound (see [70], Theorem 12.1),

$$\mathbb{P}_T(X^*_i=e^*)\le 2\sqrt{Y_u}e^{-u^2/2i}.$$

Since, by Jensen's Inequality, $E_{\mathtt{MT}}(\sqrt{Y_n})\le 1$,

$$\mathbb{P}_{\mathtt{MT}}(X_i\ge u \text{ for some } i\le t)\le 2te^{-u^2/2t}.$$

We have the following corollary, whose proof is omitted

Corollary 3.3.9

$$\mathbb{P}_{\mathtt{IMT}}(|h(X_i)| \geq u \textit{ for some } i \leq t) \leq 4t^3 e^{-(u-1)^2/2t}.$$

Proof : see [78], Corollary 2.

We can now finish the proof of the second part of Proposition 3.3.3. Under $\mathbb{P}_{\mathtt{IMT}}$, the increments $h(X_{i+1}) - h(X_i)$ are stationnary, therefore, for any ϵ and $r, s \leq t$ with $|s-r| \leq t^\delta$,

$$\mathbb{P}_{\mathtt{IMT}}(|h(X_r) - h(X_s)| \geq t^{1/2-\epsilon}) \leq \mathbb{P}_{\mathtt{IMT}}(|h(X_{r-s})| \geq t^{1/2-\epsilon}) \leq 4t^3 e^{-t^{1-\delta-2\epsilon}}.$$

Whence, by Markov's Inequality, for all t large,

$$P_{\mathtt{IMT}}\left(\mathbb{P}_T^0\left(|h(X_{r-s})| \geq t^{1/2-\epsilon}\right) \geq e^{-t^{1-\delta-\epsilon}}\right) \leq e^{-t^{1-\delta-\epsilon}}.$$

Consequently,

$$P_{\mathtt{IMT}}\left(\mathbb{P}_T^0\left(\sup_{r,s\leq t, |r-s|\leq t^\delta} |h(X_r) - h(X_s)| \geq t^{1/2-\epsilon}\right) \geq e^{-t^{1-\delta-\epsilon}}\right) \leq e^{-t^{1-\delta-\epsilon}}.$$

The Borel-Cantelli Lemma completes the proof.

We are now able to finish the proof of Theorem 3.3.1. Due to Proposition 3.3.2, the process $\{M\lfloor nt \rfloor / \sqrt{\sigma^2\eta^2 n}\}$ converges, for `IMT` almost every T, to a standard Brownian motion, as n goes to infinity. Further, by Theorem 14.4 of [11], $\{M\rho_{nt}/\sqrt{\sigma^2\eta^2 n}\}$ converges, for `IMT` almost every T, to a standard Brownian motion, as n goes to infinity. Proposition 3.3.3 implies that the sequence of processes $\{Y_t^n\} = \{h(X_{\rho_{nt}})/\sqrt{\sigma^2 n}\}$ is tight and its finite dimensional distributions converge to those of a standard Brownian motion, therefore it converges in distribution to a standard Brownian motion, and, applying again Theorem 14.4 of [11], so does $\{h(X_{\lfloor nt \rfloor}/\sqrt{\sigma^2 n}\}$.

3.4 Proof of Theorem 2.5.6.

In this section we finish the proof of Theorem 2.5.6. Our argument relies on a coupling between random walks on `MT` and on `IMT` trees, quite similar to the coupling exposed in [78]. Let us introduce some notations : for T, S two trees, finite or infinite, we set LT the leaves of T, that is the vertices of T that have no offspring, $T^o = T/LT$ and for $v \in T$ we denote by

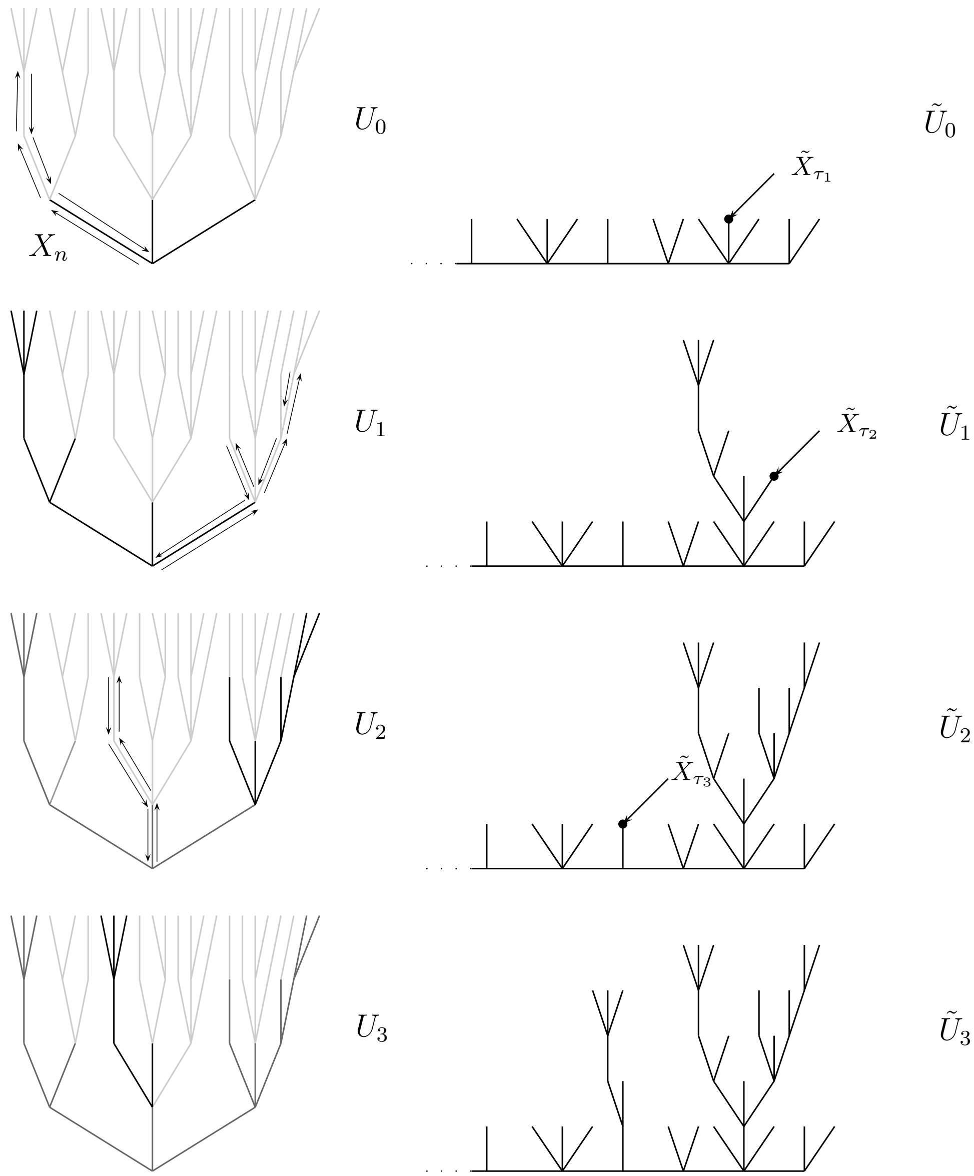


Figure 3.3: the coupling

$T \circ^v S$ the tree obtained by gluing the root of S to the vertex v of T, with vertices marked as in their original tree (the vertex coming from both v and the root of S is marked as v). Given a tree $T \in \mathbb{T}$ and a path $\{X_t\}$ on T we construct a family of finite trees T_i, U_i as follows : let $\tau_0 = \eta_0 = 0$, and U_0 the finite tree consisting of the root e of T and its offspring, marked as in T. For $i \geq 1$, let

$$\tau_i = \min\{t \geq \eta_{i-1} : X_t \in LU_{i-1}\} \tag{3.4.1}$$

$$\eta_i = \min\{t > \tau_i; X_t \in U_{i-1}^o\}. \tag{3.4.2}$$

Let T_i be the tree "explored" by the walk during the excursion $[\tau_i, \eta_i)$, that is to say T_i is composed of the vertices of T visited by $\{X_t, t \in [\tau_i, \eta_i)\}$, together with their offspring, marked as in T, and the root of T_i is X_{τ_i}. Let $U_i = U_{i-1} \circ^{X_{\tau_i}} T_i$ be the tree explored by the walk from the beginning. We call $\{u_t^i\}_{t=0}^{\eta_i - \tau_i - 1}$ the path in T_i defined by $u_t^i = X_{\tau_i + t}$. If T is distributed according to `MT`, and X_t is the path of the random walk on T, then, the walk being recurrent, $\mathbb{P}_{\mathtt{MT}}-$almost surely $T = \lim U_i$.

We are now going to construct $\tilde{T} \in \tilde{\mathbb{T}}$, a tree with a semi-infinite ray emanating from the root, coupled with T, and a path $\{\tilde{X}_t\}$ on T, in such a way that, if T is distributed according to `MT`, and X_t is the path of the random walk on T, then $\tilde{T}$ will be distributed according to `IMT` and $\{\tilde{X}_t\}$ will follow the law of a random walk on $\tilde{T}$.

Let $\tilde{U}_o$ be the tree defined as follows : we choose a vertex denoted by e, as the root of $\tilde{U}_o$, and a semi-infinite ray $\{e = v_0, v_1, ...\}$. To each vertex $v_i \in Ray$ different from e we attach independently a set of marked vertices with law $\hat{q}$. To e we attach a set of children with distribution $(q + \hat{q})/2$ If $i \geq 1$ we chose one of those vertices, with probability $\frac{A(x)}{\sum_y A(y)}$, and identify it with v_{i-1}. We obtain a tree with a semi-infinite ray and a set of children for each vertex v_i on Ray, one of them being v_{i-1}.

We set $\tilde{\tau}_0 = \tilde{\eta}_0 = 0$. Recalling the relation (3.2.1) between the A_x and the $\omega(x, y)$, one can easily check that for any vertex x, knowing the $\{w(x, y)\}_{y \in T}$ is equivalent to knowing $\{A(x_i)\}_{x_i \text{ children of } x}$. Thus, knowing $\tilde{U}_0$ one can compute the $\{\omega(x, y)\}_{x \in Ray, y \in \tilde{U}_0}$ and define a random walk $\tilde{X}_t$ on $\tilde{U}_0$, stopped when it gets off Ray. We set accordingly $\tilde{\tau}_1 = \min\{t > 0 : \tilde{X}_t \in L\tilde{U}_0\}$.

We are now going to "glue" the first excursion of $\{X_t\}$. Let

$$\begin{aligned}
\tilde{U}_1 &= \tilde{U}_0 \circ^{\tilde{X}_{\tilde{\tau}_1}} T_1, \\
\tilde{\eta}_1 &= \tilde{\tau}_1 + \eta_1 - \tau_1, \\
\{\tilde{X}_t\}_{t=\tilde{\tau}_1}^{\tilde{\eta}_1 - 1} &= u^1_{t-\tilde{\tau}_1}, \\
\tilde{X}^{\tilde{\eta}_1} &= \overleftarrow{\tilde{X}^{\tilde{\eta}_1 - 1}}.
\end{aligned}$$

One can easily check that $\{\tilde{X}_t\}_{t\leq\tilde{\eta}_1}$ follows the law of a random walk on $\tilde{U}_1$.

We iterate the process, in the following way : for $i > 1$, start a random walk $\{\tilde{X}_t\}_{t\geq\tilde{\eta}_{i-1}}$ on $\tilde{U}_{i-1}$, and define

$$\begin{aligned}
\tilde{\tau}_i &= \min\{t > 0 : \tilde{X}_t \in L\tilde{U}_{i-1}\}, \\
\tilde{U}_i &= \tilde{U}_{i-1} \circ^{\tilde{X}_{\tilde{\tau}_i}} T_i, \\
\tilde{\eta}_i &= \tilde{\tau}_i + \eta_i - \tau_i, \\
\{\tilde{X}_t\}_{t=\tilde{\tau}_i}^{\tilde{\eta}_i - 1} &= u^i_{t-\tilde{\tau}_i}, \\
\tilde{X}^{\tilde{\eta}_i} &= \overleftarrow{\tilde{X}^{\tilde{\eta}_i - 1}}.
\end{aligned}$$

Finally, set $\tilde{U} = \bigcup_0^\infty \tilde{U}_i$ and $\tilde{T}$ the tree obtained by attaching independents `MT` trees to each leaves of $\tilde{U}$. It is a direct consequence of the construction that

Proposition 3.4.1 *If T is distributed according to* `MT` *and X_t follows $\mathbb{P}_T$, then $\tilde{T}$ is distributed according to* `IMT`*, and $\tilde{X}_t$ follows $\mathbb{P}_{\tilde{T}}$.*

As a consequence, under proper assumptions on q, application of Proposition 3.3.1 implies that for `MT` almost every T the process $\{h(\tilde{X}_{\lfloor nt\rfloor})/\sqrt{\sigma^2 n}\}$ converges to a standard Brownian motion, as n goes to infinity.

We introduce $R_t = h(\tilde{X}_t) - \min_{i=1}^t h(\tilde{X}_i)$. We get immediately that $\{R_{\lfloor nt\rfloor}/\sqrt{\sigma^2 n}\}$ converges to a Brownian motion reflected to its minimum, which has the same law as the absolute value of a Brownian motion.

In order to prove Theorem 2.5.6, we need to control the distance between R_t and $|X_t|$.

Let $I_t = \max\{i : \tau_i \leq t\}$ and $\tilde{I}_t = \max\{i : \tilde{\tau}_i \leq t\}$ the number of excursions started by $\{X_t\}$ and $\{\tilde{X}_t\}$ before time t. Let $\Delta_t = \sum_{i=1}^{I_t}(\tau_i - \eta_{i-1})$ and $\tilde{\Delta}_t = \sum_{i=1}^{\tilde{I}_t}(\tilde{\tau}_i - \tilde{\eta}_{i-1})$, which measure the time spent by $\{X_t\}$ and $\{\tilde{X}_t\}$ outside the coupled excursions before time t. By construction, the distance between R_t and $|X_t|$ comes only from the parts of the walks outside those excursion. In order to control these parts, we set for $0 \le \alpha < 1/2$

$$\Delta_t^\alpha = \sum_{i=1}^{I_t} \sum_{s=\eta_{i-1}}^{\tau_i - 1} \mathbb{1}_{|X_s| \le t^\alpha};$$

similarly,

$$\tilde{\Delta}_t^\alpha = \sum_{i=1}^{\tilde{I}_t} \sum_{s=\tilde{\eta}_{i-1}}^{\tilde{\tau}_i - 1} \mathbb{1}_{d(\tilde{X}_s, Ray) \le t^\alpha}.$$

Finally, let

$$\mathbf{B}_t = \max_{0 \le r < s \le t, \tilde{X}_r \in Ray, \tilde{X}_s \in Ray} (h(\tilde{X}_s) - h(\tilde{X}_r)),$$

be the maximum amount the walk $\{\tilde{X}_t\}$ moves against the drift on Ray. We have the following

Proposition 3.4.2 *Under the assumptions of Theorem 2.5.6, for some $\alpha < 1/2$*

$$\lim_{t\to\infty} \mathbb{P}_T(\Delta_t \neq \Delta_t^\alpha) = 0, \ \mathtt{MT} - a.s., \tag{3.4.3}$$

and

$$\lim_{t\to\infty} \mathbb{P}_T(\tilde{\Delta}_t \neq \tilde{\Delta}_t^\alpha) = 0, \ \mathtt{IMT} - a.s.. \tag{3.4.4}$$

Further,

$$\limsup \frac{\Delta_t}{t} = 0, \ \mathtt{MT} - a.s., \tag{3.4.5}$$

and

$$\limsup \frac{\tilde{\Delta}_t}{t} = 0, \ \mathtt{IMT} - a.s.. \tag{3.4.6}$$

Finally,

$$\limsup \frac{\mathbf{B}_t}{\sqrt{t}} = 0, \ \mathtt{IMT} - a.s.. \tag{3.4.7}$$

Before proving this proposition, note that on the event $\{\Delta_t = \Delta_t^\alpha\} \cap \{\tilde{\Delta}_t = \tilde{\Delta}_t^\alpha\}$, we have

$$\min_{s:|s-t| \le \Delta_t + \tilde{\Delta}_t} ||X_t| - R_s| \le 2t^\alpha + \mathbf{B}_t.$$

Therefore we obtain that almost surely, there exists a time change θ_t such that, for t large enough,

$$\frac{|X_t - R_{\theta_t}|}{\sqrt{t}} \to_{t\to\infty} 0$$

and

$$\frac{|\theta_t - t|}{t} \to_{t\to\infty} 0.$$

As we said earlier, Proposition 3.3.1 implies that $\{R_{\lfloor nt \rfloor}/\sqrt{\sigma^2 n}\}$ converges, as n goes to infinity, to the law of the absolute value of a Brownian motion. Therefore so does $\{R_{\lfloor n\theta_t \rfloor}/\sqrt{\sigma^2 n}\}$. We deduce the result for $|X_t|$.

We now turn to the proof of Lemma 3.4.2. We introduce some notations: for $k \geq 1$, let $a_k = \sum_{j=1}^{k} \tau_j$, $b_k = \sum_{j=0}^{k-1} \eta_j$ and $J_k = [a_k - b_k + k, a_{k+1} - b_{k+1} + k]$. Note that $\{J_k\}_{k\geq 1}$ is a partition of $\mathbb{N}$, such that the length of J_k is equal to the time spent by the walk between the $k-th$ and the $k+1-th$ excursion. For $s \in J_k$, let $\mathbf{t}(s) = \eta_k + s - (a_k - b_k + k)$ and $Y_0 = 0, Y_1 = X_{\tau_1}$, and $Y_s = X_{\mathbf{t}(s)}$. $\{Y_s\}_{s\geq 0}$ is the walk X_n restricted off the excursions, it is clearly not Markovian, nevertheless, it is adapted to the filtration $G_s = \sigma(X_k, k \leq t(s))$. For a fixed t, we set the sequence Θ_i of stopping times with respect to G_s defined by $\Theta_0 = 0$ and

$$\Theta_i = \min\{s > \Theta_{i-1} : \big||Y_s| - |Y_{\Theta_{i-1}}|\big| = \lfloor (\log t)^{3/2} \rfloor\}.$$

Similarly, we set, for $k \geq 1$, $\tilde{a}_k = \sum_{j=1}^{k} \tilde{\tau}_j$, $\tilde{b}_k = \sum_{j=0}^{k-1} \tilde{\eta}_j$ and $\tilde{J}_k = [\tilde{a}_k - \tilde{b}_k + k, \tilde{a}_{k+1} - \tilde{b}_{k+1} + k]$, and for $s \in \tilde{J}_k$, we call $\tilde{t}(s) = \tilde{\eta}_k + s - (\tilde{a}_k - \tilde{b}_k + k)$ and $\tilde{Y}_0 = 0, \tilde{Y}_1 = \tilde{X}_{\tilde{\tau}_1}$, and $\tilde{Y}_s = \tilde{X}_{\tilde{t}(s)}$ the walk $\tilde{X}_n$ restricted off the excursions. We set $\tilde{G}_s = \sigma(\tilde{X}_k, k \leq \tilde{t}(s))$. For a fixed t, we set the sequence of stopping times $\tilde{\Theta}_i$ with respect to $\tilde{G}_s$ defined by $\tilde{\Theta}_0 = 0$ and

$$\tilde{\Theta}_i = \min\left\{ s > \tilde{\Theta}_{i-1} : \left| d(\tilde{Y}_s, Ray) - d(\tilde{Y}_{\Theta_{i-1}}, Ray)|\right| = \lfloor (\log t)^{3/2} \rfloor \right\}.$$

We need the following lemma, whose demonstration will be postponed.

Lemma 3.4.3 *For all $\epsilon > 0$*

$$\lim_{t\to\infty} \mathbb{P}_T\left(\sum_{i=1}^{t^{1/2+\epsilon}} (\eta_i - \tau_i) < t\right) = 0,\ \mathtt{MT} - a.s., \tag{3.4.8}$$

$$\lim_{t\to\infty} \mathbb{P}_T\left(\sum_{i=1}^{t^{1/2+\epsilon}} (\tilde{\eta}_i - \tilde{\tau}_i) < t\right) = 0,\ \mathtt{IMT} - a.s., \tag{3.4.9}$$

$$\exists \epsilon' > 0 : \lim_{t\to\infty} \mathbb{P}_T\left(\exists s \leq t, W_{X_s} > t^{1/4-\epsilon'}\right) = 0,\ \mathtt{MT} - a.s., \tag{3.4.10}$$

$$\textit{and } \lim_{t\to\infty} \mathbb{P}_T\left(\exists s \leq t, W_{X_s} > t^{1/4-\epsilon'}\right) = 0,\ \mathtt{IMT} - a.s., \tag{3.4.11}$$

$$\lim_{t\to\infty} \mathbb{P}_T(\exists k \leq I_t, \Theta_{i-1}, \Theta_i \in J_k, |Y_{\Theta_i}| > |Y_{\Theta_{i-1}}|) = 0,\ \mathtt{MT} - a.s., \tag{3.4.12}$$

$$\lim_{t\to\infty} \mathbb{P}_T(\exists k \leq I_t, \tilde{\Theta}_{i-1}, \tilde{\Theta}_i \in \tilde{J}_k, d(\tilde{Y}_{\tilde{\Theta}_i}, Ray) > d(\tilde{Y}_{\tilde{\Theta}_{i-1}}, Ray) = 0,\ \mathtt{IMT} - a.s., \tag{3.4.13}$$

$$\lim_{t\to\infty} \mathbb{P}_T(X_s \in \cup_{k=t^\alpha-(\log t)^2}^{t^\alpha} \mathbf{A}_k^\epsilon \textit{ for some } s \leq t) = 0,\ \mathtt{MT} - a.s., \tag{3.4.14}$$

$$\lim_{t\to\infty} \mathbb{P}_T(X_s \in \cup_{k=t^\alpha-(\log t)^2}^{t^\alpha} \mathbf{B}_k^\epsilon \textit{ for some } s \leq t) = 0,\ \mathtt{IMT} - a.s.. \tag{3.4.15}$$

Using this lemma, we can finish the proof of Proposition 3.4.2. We shall prove the following statement, which implies (3.4.3) : for some $\alpha \leq 1/2$,

$$\lim_{t\to\infty} \mathbb{P}_T(\max_{s\in\cup_{k=1}^{I_t} J_k} |Y_s| \geq t^\alpha) = 0,\ \mathtt{MT} - a.s.. \tag{3.4.16}$$

It is a direct consequence of (3.4.8) and (3.4.12) that, MT almost surely, with $\mathbb{P}_T$ probability approaching 1 as t goes to infinity,

$$t(\Theta_{2t^{1/2+\epsilon}}) > t,$$

whence, using lemma 3.4.3,

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{P}_T\big(\max_{s\in\cup_{k=1}^{I_t} J_k} |Y_s| \geq t^\alpha\big) \\
&\qquad \leq \limsup_{t\to\infty} \sum_{i=0}^{2t^{1/2+\epsilon}} \mathbb{P}_T\left(\exists j > i : |Y_{\Theta_j}| \geq t^\alpha - (\log t)^2,\ Y_{\Theta_i} = e,\right. \\
&\qquad\qquad S_{Y_{\Theta_j}} \geq (\eta - \epsilon_1)t^\alpha/2,\ |Y_{\Theta_k}| > 0, \forall i < k \leq j; \\
&\qquad\qquad \left.|S_{X_s} - |X_s|| \leq \epsilon |X_s|, \forall s \leq t\right) := \limsup_{t\to\infty} \sum_{i=1}^{2t^{1/2+\epsilon}} P_{i,t}\,;
\end{aligned}$$

where ϵ', ϵ_1 are positive numbers that can be chosen arbitrarily small.

For a fixed i and a fixed t, we set $\tilde{M}_s = S_{X_{\Theta_{i+s}}}$, and

$$K_t = \min\left\{s > 1 : X_r = 0 \text{ for some } r \in [s(\theta_{i+1}), s(\theta_{i+t})]\right\}.$$

The process $\{N_s\} = \{\tilde{M}_{s\wedge K_r} - \tilde{M}_1\}$ is a supermartingale with respect to the filtration $\tilde{G}'_s = \tilde{G}_{\theta_{i+s}}$; indeed as long as the walk does not come back to the root, the conditional expectation of $S_{Y_{s+1}} - S_{Y_s}$ is lesser or equal to 0, and by construction the walk can only return to the root at a Θ_i.

Note that M_s and N_s depend on t, whereas this is omitted in the notation. Let A_s be the predictable process such that $N_s + A_s$ is a martingale.

Note that, on the event $\{W_{X_s} \leq t^{1/4-\epsilon'}, \forall s \leq t\}$ the increments of N_s are bounded by $t^{1/4-\epsilon'}(\log t)^{3/2}$. One can easily see that the increments of A_s are also bounded by $t^{1/4-\epsilon'}(\log t)^{3/2}$. Therefore Azuma's Inequality implies

$$P_{i,t} \leq \exp\left(-t^{2\alpha}/t1/2 + 2\epsilon + 2(1/4 - \epsilon')\right).$$

Recalling that we can choose ϵ arbitrarily small and α arbitrarily close to $1/2$, we get the result.

The proof of (3.4.4) is quite similar and omitted.

To prove (3.4.5) we introduce

$$T^{\epsilon}(t) = \min\{s : |X_s| \geq t^{1/2+\epsilon}\}. \tag{3.4.17}$$

By Lemma 3.3.8, we have

$$\mathbb{P}_{\mathtt{MT}}(T_{\epsilon}(t) < t) \leq te^{-t^{2\epsilon}}.$$

Using the Borel-Cantelli Lemma, we get that, MT almost surely

$$\mathbb{P}_T(T_{\epsilon}(t) < t) \leq e^{-t^{\epsilon}} \text{ for } t > t_0(T). \tag{3.4.18}$$

Let $C_{0,l}$ be the conductance between the root and the level l of the tree. Recalling that for w an offspring of v, the conductance associated to the edge $[v, w]$ is C_w, Thomson's principle implies that

$$C_{0,l}^{-1} = \inf_{f \text{ unit flow}} \sum_{i=0}^{l} \sum_{v \in T_i} \sum_{w \text{ offspring of v}} \frac{f_{v,w}^2}{C_w}.$$

As one can easily check, $f_{v,w} = \frac{C_w W_w}{W_e}$ is a unit flow from the root to T_l, so we get

$$C_{0,l}^{-1} \leq \frac{1}{W_e} \sum_{i=1}^{l} \sum_{v \in T_i} C_w W_w^2.$$

As, conditionally to $\mathcal{G}_i$, W_w^2 are independent and identically distributed variables, with finite moment of order two (the assumption needed for that is $\kappa > 4$), we have

$$E_{\mathtt{MT}}\left[\left(\sum_{v \in T_i} C_w W_w^2 - \sum_{v \in T_i} C_w E_{\mathtt{MT}}[W_w^2]\right)^2\right] \leq C_{17}\rho(2)^i,$$

for some constant C_{17}, then, using Markov's Inequality, for every $\nu > 0$ there exists a constant C_{18} such that

$$P_{\mathtt{MT}}\left[\sum_{v \in T_i} C_w |W_w^2 - E[W_w^2]| > \nu\right] \leq C_{18}\rho(2)^i.$$

This is summable, so by the Borel-Cantelli Lemma, for some constant $C(T)$ dependant only on T, we get

$$\sum_{v \in T_i} C_w W_w^2 \leq C(T) \sum_{v \in T_i} C_w.$$

The last part being convergent, thus bounded, we get

$$C_{0,l}^{-1} \leq C(T) l. \tag{3.4.19}$$

If $L_0(t)$ denotes the number of visits to the root before time t, we get

$$\mathbb{E}_T[L_0(T_\epsilon(t))] = 1 + C_{0,t^{1/2+\epsilon}}^{-1},$$

indeed $L_0(T_\epsilon(t)) - 1$ follows a geometric law with parameter $1 - C_{0,t^{1/2+\epsilon}}^{-1}$.

Let $N_t(\alpha) = \sum_{k=0}^{t} \mathbb{1}_{|X_k| \leq t^\alpha}$ On the event that $T_\epsilon(t) > t$, we have, using Markov's property,

$$\mathbb{E}_T[N_t(\alpha); T_\epsilon(t) > t] \leq \mathbb{E}_T[L_0(T_\epsilon(t))]\pi\left(\bigcup_0^{t^\alpha} T_t\right) \leq C_{19}(T) t^{1/2+\epsilon+\alpha}.$$

Thus as $\mathbb{P}_T(T_\epsilon(t) \leq t) \leq C_{19}(T) e^{-n^\epsilon}$, using the monotonicity of $N_n(\alpha)$, we obtain $N_t(\alpha)/t \to 0$, from which the result follows, as $\Delta_t^\alpha \leq N_t^\alpha$ and $\mathbb{P}_T(\Delta_t \neq \Delta_t^\alpha) \to 0$.

Now we turn to the proof of (3.4.6). By the same calculations as in the proof of Lemma 3.3.7, for $\kappa > 5$, we get that $\mathbb{E}_{\mathtt{IMT}}[\sum_{s \leq t} \mathbb{1}_{d(X_s, Ray) < t^\alpha}] \leq t^{1/2+\alpha+\epsilon}$ for any $\epsilon > 0$, from which the result follows by an application of Markov's Inequality and the Borel-Cantelli Lemma, using also the fact that the quantity in the expectation is non-decreasing in n.

The conductance from v_k to v_{k-u} is at most $C_{v_{k-u}}$, thus we have the bound

$$\mathbb{P}_T(\mathbf{B}_t > u) \leq t \sum_{k=u}^{t} \Pi_{i=k}^{k-u} A(v_i).$$

By Theorem 3.1.1 and Lemma 3.2.2, the $\mathtt{IMT}$-expectation of the right hand side is of order at most $t^2 \rho(2)^u$, therefore (3.4.7) follows by standard arguments.

3.5 Proof of Lemma 3.4.3.

It is clear that (3.4.8) and (3.4.9) are equivalent. We postpone the proof of these parts to the end of the section.

Proof of (3.4.10) : following [78], we call "fresh time" a time where the walk explore a new vertex, we have

$$\begin{aligned}
\mathbb{P}_{\mathtt{MT}}\left(\exists s \leq t, W_{X_s} > t^{1/4-\epsilon'}\right) &\leq \sum_{0}^{t} \mathbb{P}_{\mathtt{MT}}[W_{X_s} > t^{1/4-\epsilon'}; s \text{ is a fresh time}] \\
&= \mathbb{P}_{\mathtt{MT}}[W_0 > t^{1/4-\epsilon'}] < C_{20} t / t^{\mu(1/4+\epsilon')},
\end{aligned}$$

for $\mu < \kappa$. If $\kappa > 8$, for ϵ small enough, we can chose μ such that this is summable. Then the Borel-Cantelli Lemma implies the result.

Proof of (3.4.11) We are going to use the same arguments, excepted that we have to treat separately the vertices on *Ray*. More precisely

$$\begin{aligned}
&\mathbb{P}_{\mathtt{MT}}\left(\exists s \leq t, W_{X_s} > t^{1/4-\epsilon'}\right) \\
&\qquad \leq \sum_{0}^{t} \mathbb{P}_{\mathtt{IMT}}[W_{X_s} > t^{1/4-\epsilon'}; s \text{ is a fresh time and } X_s \notin Ray] + \\
&\qquad\qquad \mathbb{P}_{\mathtt{MT}}\left(\exists s \leq t, W_{v_s} > t^{1/4-\epsilon'}\right).
\end{aligned}$$

The second term is easily bounded, and the first one is similar to the previous case.

Proof of (3.4.12) : the event in the probability in (3.4.12) implies that, before time t the walk X_s gets to some vertex u, situated at least at a distance $\lfloor(\log t)^{3/2}\rfloor$, then back to the ancestor $a(u)$ of u situated at distance $\lfloor(\log t)^{3/2}\rfloor$ from u, then back again. Decomposing on the hittings of the root, we can majorate this probability by

$$\sum_{s\le t}\mathbb{P}_T(X_t=e)\sum_{k=\lfloor(\log t)^{3/2}\rfloor}^{t}\sum_{u\in T_k}\mathbb{P}_T(H_u<H_e)\mathbb{P}_T^{a(u)}(H_u<t),$$

where H_u stands for the hitting time of u. Using the fact that the conductance from 0 to u is bounded by C_u, the probability we are considering is at most

$$n\sum_{k=\lfloor(\log t)^{3/2}\rfloor}^{t}\sum_{u\in T_k}C_u\mathbb{P}_T^{a(u)}(H_u<t).$$

Denoting by $C(v\to u)$ the conductance between v and u, we have easily

$$\mathbb{P}_T^v(H_u<t)<t\frac{C(v\to u)}{\pi(v)}<c_1t\frac{C_u}{C_v}.$$

As a direct consequence of Theorem 3.3.1, we have

$$\begin{aligned}E_{\mathtt{MT}}\left[\sum_{u\in T_k}C_u\mathbb{P}_T^{a(u)}(H_u<t)\right]&\le c_1t^2E_{\mathtt{MT}}\left[\sum_{u\in T_k}C_u\frac{C_u}{C_{a(u)}}\right]\\&\le c_1t^2\left(E_q\left[\sum A_i\exp(\log(A_i))\right]\right)^{\lfloor(\log t)^{3/2}\rfloor}\le c_1t^2\rho(2)^{\lfloor(\log t)^{3/2}\rfloor}.\end{aligned}$$

The result follows by an application of the Borel-Cantelli Lemma.

Proof of (3.4.13) : The proof is quite similar to the precedent argument, summing over the different $T^{(v_i)}$.

Proof of (3.4.14) : using $\kappa>5$, by Lemma 3.3.4 we can find an $\varepsilon>0$ such that IMT-almost surely the sequence $n^{3/2+\varepsilon}\pi(\mathbf{A}_n^\epsilon)$ is summable, thus bounded, so there exists a constant $C'(T)$ such that for each n, $C_{e\to\mathbf{A}_{n^\epsilon}}\le C'(T)/n^{3/2+\varepsilon}$. Recalling from the proof of (3.4.5) the definition of $L_0(t)$, and $T_\epsilon(t)$ we have

$$\mathbb{P}_T(X_t\in\mathbf{A}_{t^\alpha}^\epsilon;t\le T_\epsilon(t))\le\mathbb{E}_T[L_0(T_\epsilon(t))]C'(T)/t^{\alpha(3/2+\varepsilon)}\le t^{1/2+\epsilon'-\alpha(1+\varepsilon)},\tag{3.5.1}$$

where ϵ' can be chosen arbitrarily close to 0. By choosing α close enough to 1/2, the result follows easily, using (3.4.18).

Proof of (3.4.15) : we recall from (3.3.16) the definition of the sets $\mathbf{B}_n^\epsilon$. By the same argument as in the proof of Lemma 3.3.7, we get

$$\lim_{t\to\infty} \mathbb{P}_T(X_s \in \mathbf{B}_{t^\alpha}^\epsilon \text{ for some } s \le t\} \le H_{\lfloor t^{1/2+\epsilon}\rfloor} \sum_{i=0}^{\lfloor t^{1/2+\epsilon}\rfloor} U_i^{t^\alpha},$$

with $H_t = 1 + \sum_{j=0}^{t-1}\prod_{k=j-1}^{t} A(v_k)$, and $U_i^{t^\alpha}$ is the probability to get to $B_{t^\alpha}^\epsilon$ during one excursion in T^{v_i}. By the same argument as in the proof of Lemma 3.3.4, we get that, almost surely, there exists a constant $C''(T)$ such that

$$H_t \le C''(T) t^{1/7},$$

whence

$$\lim_{t\to\infty} \mathbb{P}_T(X_s \in \mathbf{B}_{t^\alpha}^\epsilon \text{ for some } s \le t\} \le C''(T) t^{1/7} \sum_{i=0}^{\lfloor t^{1/2+\epsilon}\rfloor} U_i^{t^\alpha}.$$

Then, denoting $\sum_{t=0}^{\infty} U_i^t t^{1+\epsilon'} := E_i$, the E_i are i.i.d. variables (under $\mathtt{IMT}$) with finite expectation for ϵ' small enough and $U_i^t < \frac{1}{t^{3/2}} E_i$. Then the result follows, using the law of large numbers.

Proof of (3.4.8) and (3.4.9): Note that, under $\mathtt{MT}$, the random variables $\eta_i - \tau_i$ are i.i.d.. On the other hand, as a consequence of (2.5.5), for some constant $\nu_0 > 0$,

$$\mathbb{P}_{\mathtt{MT}}[\eta_i - \tau_i > x] \ge \nu_0 \mathbb{P}_{\mathtt{MT}}[T_0 > x],$$

Where T_0 is the first return to the root. We recall from (3.4.17) that

$$T^\epsilon(t) = \min\{t : |X_s| \ge t^{1/2+\epsilon}\}$$

Then, following the proof of Lemma 10 of [78], we have,

$$\mathbb{P}_T[T_0 > t] \ge \mathbb{P}_T[T_0 > T^{\epsilon/2}(t)]\mathbb{P}_T[T^{\epsilon/2}(t) \ge t | T_0 > T^\epsilon(t)]. \tag{3.5.2}$$

As a consequence of (3.4.19), for come constant depending on the tree $C_3(T)$,

$$\mathbb{P}_T[T_0 > T^{\epsilon/2}(t)] > C_3(T) t^{-1/2-\epsilon/2}.$$

On the other hand,

$$\mathbb{P}_T[T^{\epsilon/2}(t) < t | T_0 > T^{\epsilon/2}(t)] \le \frac{\mathbb{P}_T[T^{\epsilon/2}(t) < t]}{\mathbb{P}_T[T_0 > T^{\epsilon/2}(t)]} \le C_4(T) t^{1/2+\epsilon} e^{-t^{\epsilon/2}},$$

$\mathtt{MT}-$almost surely, using (3.4.18) and the previous estimate. We get then that almost surely, for t large enough (the "enough" depending on T),

$$\mathbb{P}_T[T^{\epsilon/2}(t) > n | T_0 > T^{\epsilon/2}(t)] > 1/2.$$

Therefore for some positive constant $C_5(T)$,

$$\mathbb{P}_T[T_0 > t] \geq C_5(T) t^{-1/2-\epsilon/2}.$$

We deduce by taking the expectation that

$$\mathbb{P}_{\mathtt{MT}}[T_0 > t] \geq C_{22} t^{-1/2-\epsilon/2},$$

for some positive and deterministic constant C_{22}. Now

$$\mathbb{P}_{\mathtt{MT}}\left[\sum_{i=1}^{t^{1/2+\epsilon}} \eta_i - \tau_i < t\right] \leq \left(1 - \nu_0 C_{22} t^{-1/2-\epsilon/2}\right)^{t^{1/2+\epsilon}} \leq e^{-C_{23} t^{\epsilon/2}}.$$

An application of the Borel-Cantelli Lemma finishes the proof of (3.4.8) and (3.4.9). This finishes the proof of Lemma 3.4.3.

We now turn to our last part, namely the annealed central limit theorem. The proof has many parts in common with the proof in the quenched case, so we feel free to refer to the previous part.

3.6 Proof of Theorem 2.5.7.

We recall from section 3.2 the definition of the "environment seen from the particle $T_t = \theta^v(T)$. As for the quenched case, we will first show a central limit theorem on $\mathtt{IMT}$ trees, then in a second part we will use the coupling to deduce the result for $\mathtt{MT}$ trees

3.6.1 The annealed CLT on $\mathtt{IMT}$ trees

We will first show the following proposition :

Proposition 3.6.1 *Suppose $N(e) \geq 1$, $q-a.s.$, (2.5.5). If $p = 1$, $\rho'(1) < 0$ and $\kappa \in (2, \infty]$, then there is a deterministic constant $\sigma > 0$ such that, under $\mathbb{P}_{\mathtt{IMT}}$, the process $\{h(X_{\lfloor nt \rfloor})/\sqrt{\sigma^2 n}\}$ converges in law to a standard Brownian motion, as n goes to infinity.*

Remark : This result is of great theoretical interest, as it is the only context where we are able to cover the whole case $\kappa > 2$, we could actually make the proof of Theorem 2.5.7 without this proposition, but as it has an interest in itself, we give the proof in the general case.

Proof : Let, as in the quenched setting, $0 < \delta < 1$ and ρ_t be a random variable, independent of the walk, uniformly chosen in $[t, t + t^\delta]$. We recall from (3.3.1) the definition of $S_x, x \in T$ and from (3.3.2) the definition of η. We are going to show the following

Lemma 3.6.2 *Under the assumptions of Theorem 2.5.7,*

$$\frac{S_{X_{\rho_t}}}{h(X_{\rho_t})} \to \eta, \tag{3.6.1}$$

in probability.

We admit for the moment this lemma and finish the proof of Proposition 3.6.1. We have

$$h(X_{\rho_t}) = \frac{h(X_{\rho_t})}{S_{X_{\rho_t}}} S_{X_{\rho_t}} = \eta S_{X_{\rho_t}} + \left[\frac{h(X_{\rho_t})}{S_{X_{\rho_t}}} - \eta \right] S_{X_{\rho_t}}.$$

The first term converges to a Brownian motion with variance σ, by the same arguments as in the quenched setting, while the second one is a $o(S_{X_{\rho_t}})$. The result then follows easily, using the same arguments as in the proof of Theorem 2.5.6.

We now turn to the proof of Lemma 3.6.2. The proof is quite similar to the proof of Proposition 3.3.3: we take some small $\epsilon > 0$, then we estimate the number of visits to the points in B_n^ϵ during one excursion in T^{v_i}, and estimate the number of such excursion before time n. We rely on the following lemma, similar to Lemma 3.3.4

Lemma 3.6.3 *Suppose that the assumptions of Theorem 2.5.7 are true. Then for* $1 < \lambda < \kappa - 1 \wedge 2$, *and* $n > 0$, *there exists some constant* C_1' *such that*

$$E_{\mathtt{MT}} \left[\sum_{x \in A_n^\epsilon} C_x \right] < C_1' n^{-(\lambda - 1)}.$$

Proof : the proof relies on the same ideas as the proof of Proposition 3.3.4. First recall that, for n large enough,

$$E_{\mathtt{MT}} \left[\sum_{x \in \mathbf{A}_n^\epsilon} C_x \right] \le P_{\widehat{\mathtt{MT}^*_\mathtt{n}}} \left[\left| S_{v_n} - E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[S_{v_n} | \tilde{F}_n^*] \right| > \frac{n\epsilon}{4} \right]$$
$$+ P_{\widehat{\mathtt{MT}^*_\mathtt{n}}} \left[\left| E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[S_{v_n} | \tilde{F}_n^*] - E_{\widehat{\mathtt{MT}^*_\mathtt{n}}}[S_{v_n}] \right| > \frac{n\epsilon}{4} \right] := P_1 + P_2.$$

To bound P_1, we recall that, under the law $\widehat{\mathtt{MT}^*_{\mathtt{n}}}$,

$$S_{v_n} - E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[S_{v_n}|\tilde{F}^*_n] = \sum_{i=0}^{n} \tilde{W}^*_i B_i,$$

where W_i are centered and independent random variables with bounded moments of order $\lambda + 1$ and

$$B_j = \sum_{k=0}^{j} \prod_{i=k+1}^{j} A_i.$$

Using Inequality 2.6.20 from page 82 of [80], we obtain that, for some constant C_2

$$E\left[\left|S_{v_n} - E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[S_{v_n}|\tilde{F}^*_n]\right|^{\lambda}\right] < C_2 \sum_{k=0}^{n} E[B_k^{\lambda}].$$

Then, using the same arguments as in the proof of Proposition 3.3.4, we get that $E[B_k^{\lambda}]$ is bounded independently of n and k, whence

$$E\left[\left|S_{v_n} - E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[S_{v_n}|\tilde{F}^*_n]\right|^{\lambda}\right] < C_3 n.$$

Using Markov's Inequality, there exists C_4 such that

$$P_1 < \frac{C_4}{2} n^{-(\lambda-1)}. \tag{3.6.2}$$

On the other hand, recalling (3.3.10),

$$\left|E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[S_{v_n}|\tilde{F}^*_n] - E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[S_{v_n}]\right| < C_5 + \left|\sum_{k=1}^{n} \tilde{A}_k D_k g(A_{n+1})(1 + \rho + \rho^2 + \ldots \rho^{k-1})\right|,$$

where C_5 is a finite constant and

$$D_k = \sum_{j=k+1}^{n} \prod_{i=k+1}^{j} A_i g(A_{j+1}),$$

where g is a bounded function. We recall that

$$N_k := \sum_{j=n-k}^{n} \tilde{A}_j D_j (1 + \rho + \rho^2 + \ldots \rho^{j-1})$$

is a martingale with respect to the filtration $\mathcal{H}_k = \sigma(A_j, n-k \le j \le n)$, whence, using Burkholder's Inequality,

$$E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}[(N_n)^{\lambda}] \le C_6 E_{\widehat{\mathtt{MT}^*_{\mathtt{n}}}}\left[\left(\sum_{i=0}^{n} (D_i)^2\right)^{\lambda/2}\right].$$

We recall that $1 < \lambda < (\kappa - 1) \wedge 2$, whence, by concavity, the last expression is bounded above by

$$C_6 E_{\widehat{\mathtt{MT}^*_n}} \left[\sum_{i=0}^{n} (D_i)^\lambda \right] < C_7 n.$$

Therefore, using Markov's Inequality, we get that

$$P_2 < n^{1-\lambda}.$$

This, together with (3.6.2), finishes the proof of Lemma 3.6.3.

We now finish the proof of Lemma 3.6.2. Let us go back to IMT trees. We recall the definition of the sets $\mathbf{B}_n^\epsilon$:

$$\mathbf{B}_n^\epsilon = \left\{ v \in T, d(v, \mathtt{Ray}) = n, \left| \frac{S_v^{\mathtt{Ray}}}{n} - \eta \right| > \epsilon \right\}. \tag{3.6.3}$$

We are going to prove that

$$\lim_{t\to\infty} \mathbb{P}_T(X_{\rho_t} \in \cup_{n=1}^\infty \mathbf{B}_m^\epsilon) = 0, \; \mathtt{IMT} - a.s..$$

We introduce $\gamma > 1/2$, and recall the definition of the event

$$\Gamma_t = \{\exists u \leq 2t | X_u = v_{\lfloor t^\gamma \rfloor}\}.$$

It is easy to see, using the same arguments as in the proof of Lemma 3.3.7, that

$$P_{\mathtt{IMT}}[\Gamma_t] \underset{t\to\infty}{\to} 0.$$

Furthermore, we introduce the event

$$\Gamma'_t = \{\exists\, 0 \leq u \leq t, d(X_u, Ray) > n^\gamma\};$$

then it is a direct consequence of Lemma 3.3.8 that

$$P_{\mathtt{IMT}}[\Gamma'_t] \underset{t\to\infty}{\to} 0.$$

As for the quenched case, we have

$$\begin{aligned} \mathbb{P}_{\mathtt{IMT}}(X_{\rho_t} \in \cup_{m=1}^\infty \mathbf{B}_m^\epsilon) &\leq \mathbb{P}_{\mathtt{IMT}}(X_{\rho_t} \in \cup_{m=1}^{n^\gamma} \mathbf{B}_m^\epsilon; \Gamma_t^c \cap {\Gamma'}_t^c) + \mathbb{P}_{\mathtt{IMT}}(\Gamma_t) + \mathbb{P}_{\mathtt{IMT}}({\Gamma'}_t) \\ &\leq \frac{1}{\lfloor t^\delta \rfloor} E_{\mathtt{IMT}} \left[\mathbb{E}_T \left[\sum_{s=0}^{H_{v_{\lfloor t^\gamma \rfloor}}} \mathbb{1}_{X_s \in \cup_{m=1}^{t^\gamma} \mathbf{B}^\epsilon{}_m} \right] \right] + o(1), \end{aligned} \tag{3.6.4}$$

where $H_{v_{\lfloor t^\gamma \rfloor}}$ is the first time the walk hits $v_{\lfloor t^\gamma \rfloor}$.

We recall that $T^{(v_i)}$ the subtree constituted of the vertices $x \in T$ such that $v_i \leq x, v_{i-1} \not\leq x$, the same computations as in the proof of Lemma 3.3.7 imply

$$\mathbb{P}_{\mathtt{IMT}}(X_{\rho_t} \in \cup_{m=1}^{\infty} \mathbf{B}_m^{\epsilon}) \leq \frac{1}{\lfloor t^\delta \rfloor} E_{\mathtt{IMT}} \left[\sum_{i=0}^{\lfloor t^\gamma \rfloor} \mathbb{E}_T \left[\sum_{s=0}^{H_{v_{\lfloor t^\gamma \rfloor}}} \mathbb{1}_{X_s = v_i} \right] \tilde{N}_i \right], \tag{3.6.5}$$

Where $\tilde{N}_i$ is the $\mathbb{P}_T-$expectation of the number of visits to $\cup_{m=1}^{n^\delta} \mathbf{B}_m^\epsilon \cap T^{(v_i)}$ during one excursion in $T^{(v_i)}$. Lemma 3.6.3, and the method of 3.3.5 imply that, under IMT conditioned on $\{Ray, A(v_i)\}$, $\tilde{N}_i$ are independent and identically distributed variables, with expectation at most equal to $C_1' \sum_{i=0}^{n^\gamma} i^{1-\lambda}$ for some $\lambda > 1$. By choosing γ close enough to 0, we get $E_{\mathtt{IMT}}[\tilde{N}_i | \{Ray, A(v_i)\}] \leq C_1' n^{1/2-\varepsilon}$ for some $\varepsilon > 0$. We recall that

$$\mathbb{E}_T \left[\sum_{s=0}^{H_{v_{\lfloor t^\gamma \rfloor}}} \mathbb{1}_{X_s = v_i} \right] \leq C'' \left(1 + \sum_{j=0}^{\lfloor t^\gamma \rfloor - 1} \prod_{k=j-1}^{\lfloor t^\gamma \rfloor} A(v_k) \right).$$

The latter expression has bounded expectation under IMT, as an easy consequence of Statement 3.3.1 and Lemma 3.2.2.

We deduce that

$$\mathbb{P}_{\mathtt{IMT}}(X_{\rho_t} \in \cup_{m=1}^{\infty} \mathbf{B}_m^{\epsilon}) \leq C_5 n^{\frac{1}{2} - \varepsilon + \gamma - \delta}.$$

Since γ can be chosen as close to 1/2 as needed, the exponent can be taken lower than 0. The end of the proof is then completely similar to the quenched case.

3.6.2 The annealed CLT on MT trees.

We now turn to the proof of Theorem 2.5.7. We use the coupling and the notations presented in section 3.4. Our main proposition in this part will be the following:

Proposition 3.6.4 *Under the assumptions of Theorem 2.5.7, for some $\alpha < 1/2$*

$$\lim_{t \to \infty} \mathbb{P}_{\mathtt{MT}}(\Delta_t \neq \Delta_t^\alpha) = 0, \tag{3.6.6}$$

and

$$\lim_{t \to \infty} \mathbb{P}_{\mathtt{IMT}}(\tilde{\Delta}_t \neq \tilde{\Delta}_t^\alpha) = 0. \tag{3.6.7}$$

Further, under MT,

$$\limsup \frac{\Delta_t}{t} = 0, \tag{3.6.8}$$

and under IMT,

$$\limsup \frac{\tilde{\Delta}_t}{t} = 0. \tag{3.6.9}$$

Finally, under IMT,

$$\limsup \frac{\mathbf{B}_t}{\sqrt{t}} = 0. \tag{3.6.10}$$

(Here $\limsup$ *denotes the limit in law.)*

Before proving the latter proposition, we introduce some technical estimates, whose proof will be postponed.

Lemma 3.6.5 *For all* $\epsilon > 0$

$$\lim_{t\to\infty} \mathbb{P}_{\mathtt{MT}} \left(\sum_{i=1}^{t^{1/2+\epsilon}} (\eta_i - \tau_i) < t \right) = 0, \tag{3.6.11}$$

$$\lim_{t\to\infty} \mathbb{P}_{\mathtt{IMT}} \left(\sum_{i=1}^{t^{1/2+\epsilon}} (\tilde{\eta}_i - \tilde{\tau}_i) < t \right) = 0, \tag{3.6.12}$$

$$\lim_{t\to\infty} \mathbb{P}_{\mathtt{MT}} (\exists k \le I_t, \Theta_{i-1}, \Theta_i \in J_k, |Y_{\Theta_i}| > |Y_{\Theta_{i-1}}|) = 0, \tag{3.6.13}$$

$$\lim_{t\to\infty} \mathbb{P}_{\mathtt{IMT}} (\exists k \le I_t, \tilde{\Theta}_{i-1}, \tilde{\Theta}_i \in \tilde{J}_k, d(\tilde{Y}_{\tilde{\Theta}_i}, Ray) > d(\tilde{Y}_{\tilde{\Theta}_{i-1}}, Ray) = 0, \tag{3.6.14}$$

$$\lim_{t\to\infty} \mathbb{P}_{\mathtt{MT}} (X_s \in \cup_{k=t^\alpha-(\log t)^2}^{t^\alpha} \mathbf{A}_t^\epsilon \textit{ for some } s \le t\} = 0, \tag{3.6.15}$$

$$\lim_{t\to\infty} \mathbb{P}_{\mathtt{IMT}} (X_s \in \cup_{k=t^\alpha-(\log t)^2}^{t^\alpha} \mathbf{B}_k^\epsilon \textit{ for some } s \le t) = 0. \tag{3.6.16}$$

$$\lim_{t\to\infty} \mathbb{P}_{\mathtt{MT}} \left(W_{X_s} > t^{1/4-\varepsilon} \textit{ for some } 0 \le s \le t \right) = 0 \tag{3.6.17}$$

$$\lim_{t\to\infty} \mathbb{P}_{\mathtt{IMT}} \left(W_{X_s} > t^{1/4-\varepsilon} \textit{ for some } 0 \le s \le t \right) = 0 \tag{3.6.18}$$

We now turn to the proof of (3.6.6). As a consequence of (3.6.11) and (3.6.13) that, with $\mathbb{P}_{\mathtt{MT}}$ probability approaching 1 as n goes to infinity,

$$\mathbf{t}(\Theta_{2t^{1/2+\epsilon}}) > t,$$

whence, using Lemmas 3.6.5 and 3.3.8,

$$\lim_{t\to\infty} \mathbb{P}_{\mathtt{MT}}\left(\max_{s\in\cup_{k=1}^{I_t} J_k} |Y_s| \geq t^\alpha\right)$$
$$\leq \limsup_{t\to\infty} \sum_{i=0}^{2t^{1/2+\epsilon}} \mathbb{P}_{\mathtt{MT}}\big(\exists j > i : |Y_{\Theta_j}| \geq t^\alpha - (\log t)^2,\, Y_{\Theta_i} = e,$$
$$S_{Y_{\Theta_j}} \geq (\eta - \epsilon_1)t^\alpha/2,\, |Y_{\Theta_k}| > 0, \forall i < k \leq j; W_{X_s} \leq t^{1/4-\varepsilon}\forall 0 \leq s \leq t$$
$$|S_{X_s} - |X_s|| \leq \epsilon t^{1/4-\epsilon'}|X_s|, \forall s \leq t\big) := \limsup_{t\to\infty} \sum_{i=1}^{2t^{1/2+\epsilon}} \mathbb{P}_{i,t}\,;$$

where ϵ, ϵ_1 are positive numbers that can be chosen arbitrarily small. We recall that the process $\{N_s\} = \{S_{X_{\theta_{i+s\wedge K_t}}}\}$ is a supermartingale. and that there exists a previsible and non-decreasing process A_s such that $N_s + A_s$ is a martingale. Furthermore, on the event $\{W_{X_s} \leq t^{1/4-\varepsilon}\forall 0 \leq s \leq t\}$, the increments of this martingale are bounded above by $t^{1/2-\varepsilon}$. Azuma's Inequality implies the result, as in the quenched case.

The proof of (3.6.7) is similar and omitted.

We recall that in the proofs of (3.4.5),(3.4.6) and (3.4.7) we only used the assumption $\kappa > 5$, therefore the proof of (3.6.8),(3.6.9) and (3.6.10) are direct consequence, by dominated convergence.

We now turn to the proof of Lemma 3.6.5. The proofs of (3.6.11), (3.6.12), (3.6.13), (3.6.14) and (3.6.15) follow directly from equations (3.4.8), (3.4.9), (3.4.12), (3.4.13) and (3.4.14), whose proofs did not use any assumption other than $\kappa > 5$, by dominated convergence.

To prove (3.6.16), note that, similarly to the proof of 3.6.2,

$$\mathbb{P}_{\mathtt{IMT}}(X_s \in \cup_{k=t^\alpha-(\log t)^2}^{t^\alpha} \mathbf{B}_k^\epsilon \text{ for some } s \leq t) = E_{\mathtt{IMT}}\left[\sum_{i=0}^{\lfloor t^\gamma \rfloor} \mathbb{E}_T\left[\sum_{s=0}^{H_{v_{\lfloor t^\gamma \rfloor}}} \mathbb{1}_{X_s=v_i}\right] N_i'\right],$$

where N_i' is the $\mathbb{P}_T-$expectation of the number of visits to $\cup_{k=t^\alpha-(\log t)^2}^{t^\alpha} \mathbf{B}_k^\epsilon \cap T^{(v_i)}$ during one excursion in $T^{(v_i)}$. Lemma 3.3.4 and the method of Lemma 3.3.5 imply that, under $\mathtt{IMT}$

conditioned on $\{Ray, A(v_i)\}$, N'_i are independent and identically distributed variables, up to a bounded constant, with expection at most equal to $C'(\log t)^2 t^{-\alpha(\lambda-1)}$ for some $\lambda > 2$. We also recall that

$$\mathbb{E}_T\left[\sum_{s=0}^{H_{v_{\lfloor t^\gamma \rfloor}}} \mathbb{1}_{X_s=v_i}\right] \leq C''\left(1+\sum_{j=0}^{\lfloor t^\gamma \rfloor-1}\prod_{k=j-1}^{\lfloor t^\gamma \rfloor} A(v_k)\right).$$

has bounded expectation under `IMT`, as an easy consequence of Statement 3.3.1 and Lemma 3.2.2. By choosing γ close enough to 0 and α close to 1, we get the result.

The proofs of (3.6.17) and (3.6.18) are easily deduced from the proofs of (3.4.10) and (3.4.11), the only difference being that we do not need to apply the Borel-Cantelli Lemma.

Chapter 4

The slow regime.

In this chapter we present the proofs of theorems 2.5.4 and 2.5.5 This proof is extracted from an article written in collaboration with Y. Hu and Z. Shi, and some notations are a bit different from the introduction.

Let $\mathbb{T}$ be a supercritical Galton–Watson tree rooted at $\varnothing$. Two vertices x and y are said to be connected, and denoted by $x \sim y$, if x is either the parent or a child of y. For a vertex $x \in \mathbb{T}$, we denote by $|x|$ the distance between x and the root $\varnothing$, and $\varnothing = x_0, x_1, \ldots, x_{|x|}$ the shortest path between the root and x. For each vertex $x \in \mathbb{T} \backslash \{\varnothing\}$, we denote its parent by $\overleftarrow{x}$, and its children by $(x^{(1)}, \cdots, x^{(N(x))})$, where $N(x)$ stands for the number of children of x. We will call $\mathbf{P}$ the law of the environmen, P_W the quenched law and $\mathbb{P}$ the annealed law.

In the following part we present an associated branching random walk, as well as a rapid explanation for the $(\log n)^3$ behavior.

4.0.3 Branching random walks and maxima along rays

The influence of the random environment on the behavior of (X_n) is best formulated in terms of an associated potential process. To make the presentation easier, we artificially add a special vertex, $\overleftarrow{\varnothing}$, which is to be thought of as the parent of $\varnothing$. Since the values of the transition probabilities at a finite number of vertices have no influence on any of the results of the paper, we feel free to modify the value of $\omega(\varnothing, \bullet)$, the transition probability at $\varnothing$, in such a way that $(A_i(x), 1 \le i \le N(x))$, for $x \in \mathbb{T}$ (including $x = \varnothing$ now), form an i.i.d. collection of random variables. Let $\omega(\overleftarrow{\varnothing}, \varnothing) = 1$.

The potential process associated with the random environment is defined by $V(\varnothing) := 0$

and

$$V(x) := -\sum_{y\in]\!]\varnothing,\, x]\!]} \log \frac{\omega(\overleftarrow{y},\, y)}{\omega(\overleftarrow{y},\, \overleftarrow{\overleftarrow{y}})}, \qquad x \in \mathbb{T}\backslash\{\varnothing\}, \tag{4.0.1}$$

where $\overleftarrow{\overleftarrow{y}}$ is the parent of $\overleftarrow{y}$, $[\![\varnothing,\, x]\!]$ the set of vertices on the shortest path connecting $\varnothing$ to x, and $]\!]\varnothing,\, x]\!] := [\![\varnothing,\, x]\!]\backslash\{\varnothing\}$.

Clearly, $(V(x),\, x \in \mathbb{T})$ is a branching random walk, in the usual sense of Biggins [8]. It can be described as follows: Initially, a single particle is located at the origin, which is the ancestor of the system. At time 1, the ancestor dies, giving birth to a certain number of new particles who form the first generation, and who are positioned according to the distribution of $(-\log A_i(\varnothing),\, 1 \le i \le N(\varnothing))$. At time 2, each of the particles in the first generation dies, giving birth to new particles that are positioned (with respect to their birth places) according to the same distribution of $(-\log A_i(\varnothing),\, 1 \le i \le N(\varnothing))$; these new particles form the second generation. The system goes on according to the same mechanism. We assume that for any n, each particle at generation n produces new particles independently of each other and of everything up to the n-th generation. The positions of the particles in the n-th generation are denoted by $(V(x),\, |x| = n)$.

Condition $\inf_{t\in[0,\,1]} \psi(t) = 0$ is equivalent to $\inf_{t\in[0,\,1]} \mathbf{E}(\sum_{|x|=1} \mathrm{e}^{-tV(x)}) = 1$, whereas $\psi'(1) \ge 0$ means $\mathbf{E}(\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}) \le 0$.

In the recurrent case, there is a simple relationship between the potential $(V(x),\, x \in \mathbb{T})$ and the walk (X_n). For any $k \ge 0$, let

$$\tau_k := \inf\{j \ge 1 : |X_j| = k\}, \qquad \inf\varnothing := \infty.$$

So τ_0 is the first *return* time to the root if the walk starts from $\varnothing$. It turns out that there exists $0 < c(\omega) < \infty$ possibly depending on the environment, such that for any $n \ge 1$,

$$\varrho_n := P_\omega\{\tau_n < \tau_0\} \ge \frac{c(\omega)}{n} \exp\Big(-\min_{|x|=n} \overline{V}(x)\Big), \tag{4.0.2}$$

where, for any vertex x, we write

$$\overline{V}(x) := \max_{y\in]\!]\varnothing,\, x]\!]} V(y). \tag{4.0.3}$$

Inequality (4.0.2) was proved in [44] under the additional conditions that N is deterministic and that the law of A_i does not depend on i. Since the proof is simple, we reproduce it here: For any $x \in \mathbb{T}$, let $T(x) := \inf\{i \geq 0 : X_i = x\}$ be the first hitting time of the walk at vertex x. By definition, for any $n \geq 1$, $\tau_n = \min_{|x|=n} T(x)$, so that

$$P_\omega\{\tau_n < \tau_0\} \geq \max_{|x|=n} P_\omega\{T(x) < \tau_0\}. \tag{4.0.4}$$

We fix a vertex x with $|x| = n$. To compute $P_\omega\{T(x) < \tau_0\}$, we define a random sequence $(\sigma_j)_{j\geq 0}$ (depending on x) by $\sigma_0 := 0$ and

$$\sigma_j := \inf\Big\{k > \sigma_{j-1} : X_k \in [\![\varnothing,\, x]\!] \backslash \{X_{\sigma_{j-1}}\}\Big\}, \qquad j \geq 1.$$

If the walk (X_n) is recurrent, then (σ_j) is well-defined.

Let $Z_k := X_{\sigma_k}$, $k \geq 0$, which is the restriction of (X_j) on the path $[\![\varnothing,\, x]\!]$. For $i \leq n$, let x_i be the unique vertex in $[\![\varnothing,\, x]\!]$ with $|x_i| = i$ (in particular, $x_0 = \varnothing$, $x_n = x$). Then for $1 \leq i < n$,

$$P_\omega\Big\{Z_{k+1} = x_{i+1} \,\Big|\, Z_k = x_i\Big\} = \frac{\omega(x_i,\, x_{i+1})}{\omega(x_i,\, x_{i+1}) + \omega(x_i,\, x_{i-1})} = 1 - P_\omega\Big\{Z_{k+1} = x_{i-1} \,\Big|\, Z_k = x_i\Big\},$$

which yields

$$\begin{aligned} P_\omega\{T(x) < \tau_0\} &= \omega(\varnothing, x_1)\, P_\omega\Big\{(Z_k) \text{ hits } x \text{ before hitting } \varnothing \,\Big|\, Z_0 = x_1\Big\} \\ &= \frac{\omega(\varnothing, x_1)\, \mathrm{e}^{V(x_1)}}{\sum_{z \in \,]\!]\varnothing,\, x]\!]} \mathrm{e}^{V(z)}}, \end{aligned} \tag{4.0.5}$$

the second identity following from a general formula (Zeitouni [99], formula (2.1.4)) for the exit problem of one-dimensional random walk in random environment. Going back to (4.0.4), we immediately obtain (4.0.2) with $c(\omega) := \min_{|x|=1}[\omega(\varnothing, x)\, \mathrm{e}^{V(x)}] > 0$.

The probability ϱ_n is closely related to the maximal displacement of the branching random walk. The following simple observation was implicitly stated in [44] (pp. 1993–1996):

Fact 4.0.6 *Assume* $\inf_{t\in[0,1]} \psi(t) = 0$ *and* $\psi'(1) \geq 0$. *Let* $0 < c < \infty$ *be a constant. Almost surely on the set of non-extinction,*

(i) *if* $\varrho_n \geq \mathrm{e}^{-(c+o(1))n^{1/3}}$ *for all sufficiently large* n, *then*

$$\liminf_{n\to\infty} \frac{1}{(\log n)^3} \max_{0\leq k\leq n} |X_k| \geq \frac{1}{c^3};$$

(ii) *if* $\varrho_n \leq \mathrm{e}^{-(c+o(1))n^{1/3}}$ *for all sufficiently large* n, *then*

$$\limsup_{n\to\infty} \frac{1}{(\log n)^3} \max_{0\leq k\leq n} |X_k| \leq \frac{1}{c^3}.$$

As such, an upper bound for $\min_{|x|=n} \overline{V}(x)$ yields, via inequality (4.0.2), a lower bound for ϱ_n, which, in turn, will lead to a lower bound for the maximal displacement of the walk (X_j).

Theorem 4.0.7 *Assume* $\inf_{t\in[0,\,1]} \mathbf{E}\{\sum_{|x|=1} \mathrm{e}^{-tV(x)}\} = 1$ *and* $\mathbf{E}\{\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\} \le 0$. *Let* $\theta \in (0,\,1]$ *be such that* $\mathbf{E}\{\sum_{|x|=1} V(x)\mathrm{e}^{-\theta V(x)}\} = 0$. *We have, on the set of non-extinction,*

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \min_{|x|=n} \overline{V}(x) = \Big(\frac{3\pi^2\sigma_\theta^2}{2}\Big)^{1/3}, \qquad \mathbf{P}\text{-a.s.,}$$

where

$$\sigma_\theta^2 := \frac{1}{\theta}\mathbf{E}\Big\{ \sum_{|x|=1} V(x)^2 \mathrm{e}^{-\theta V(x)}\Big\}.$$

We mention that Fang and Zeitouni [32] have independently obtained Theorem 4.0.7, under the condition that N is non-random and $A_i(\varnothing)$, for $1 \le i \le N$, are i.i.d.

Comparing Theorem 4.0.7 with Theorems 2.5.4 and 2.5.5, we observe that (4.0.2) is optimal in the case $\psi'(1) > 0$ (or, equivalently, $\mathbf{E}\{\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\} < 0$), but not in the case $\psi'(1) = 0$ (or, equivalently, $\mathbf{E}\{\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\} = 0$).

The proofs of the theorems are organized as follows.

- Section 4.1: Theorem 4.0.7.
- Section 4.3: Theorem 2.5.5.
- Section 4.4: Theorem 2.5.4, upper bound.
- Section 4.5: Theorem 2.5.4, lower bound. [This is the heart of the paper.]

Section 4.2 is devoted to a probability estimate for one-dimensional random walks, which will be exploited in the proofs of Theorems 2.5.4 and 2.5.5 later on.

Throughout the paper, we use the convention $\sum_\varnothing := 0$, $\max_\varnothing := 0$ and $\min_\varnothing := \infty$. The letter c, with or without subscript, denotes a finite and positive constant, whose value may vary from line to line. Furthermore, $a_n \sim b_n$, $n \to \infty$, means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

4.1 Proof of Theorem 4.0.7

Assume $\psi(1) = 0$, i.e., $\mathbf{E}\{\sum_{|x|=1} \mathrm{e}^{-V(x)}\} = 1$.

The condition $\mathbf{E}(N^{1+\delta}) < \infty$ in (2.5.1) guarantees that $\mathbf{P}\{N(x) < \infty,\ \forall x\} = 1$ ($N(x)$ being the number of children of x). Recall that given a vertex $x \in \mathbb{T}$, $x_0 := \varnothing$, x_1, $\cdots$,

$x_{|x|} := x$ are the vertices on $[\![\varnothing,\, x]\!]$ with $|x_i| = i$ for any $0 \le i \le |x|$. We have, for any $n \ge 1$ and any measurable function $F : \mathbb{R}^n \times \mathbb{R}^n \to [0,\, \infty)$,

$$\mathbf{E}\Big(\sum_{|x|=n} \mathrm{e}^{-V(x)} F[V(x_i),\; N(x_{i-1}),\; 1 \le i \le n] \Big) = \mathbf{E}\Big(F[S_i,\; \nu_{i-1},\; 1 \le i \le n] \Big), \tag{4.1.1}$$

where $(S_i - S_{i-1},\; \nu_{i-1})$, for $i \ge 1$, are i.i.d. random vectors, whose common distribution is determined by

$$\mathbf{E}[f(S_1,\; \nu_0)] = \mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-V(x)} f(V(x),\; N(\varnothing)) \Big) = \mathbf{E}\Big(\sum_{i=1}^{N} A_i f(-\log A_i,\; N) \Big), \tag{4.1.2}$$

for any measurable function $f : \mathbb{R}^2 \to [0,\, \infty)$. Considering only the first argument, (4.1.1) says that for any $n \ge 1$ and any measurable function $F : \mathbb{R}^n \to [0,\, \infty)$,

$$\mathbf{E}\Big(\sum_{|x|=n} \mathrm{e}^{-V(x)} F(V(x_i),\; 1 \le i \le n) \Big) = \mathbf{E}[F(S_i,\; 1 \le i \le n)], \tag{4.1.3}$$

with the distribution of S_1 determined by

$$\mathbf{E}(f(S_1)) = \mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-V(x)} f(V(x)) \Big),$$

for any measurable function $f : \mathbb{R} \to [0,\, \infty)$. Formula (4.1.3) is well-known, and can be proved by means of a simple argument by induction in n. See, for example, Biggins and Kyprianou [10]. The proof of (4.1.1) follows exactly from the same argument. In Section 4.5, we will see an extension of (4.1.1), which, in particular, gives a probabilistic interpretation of the new random walk (S_i).

[The distribution of S_1 is well-defined upon the assumption $\psi(1) = 0$. If furthermore $\psi'(1) = 0$, then $\mathbf{E}(S_1) = 0$; in words, (S_n) is a mean-zero random walk, with $\sigma^2 = \mathbf{E}(S_1^2)$.]

Formula (4.1.3) naturally leads to studying the one-dimensional random walk (S_n). However, we sometimes need to work in a slightly more general setting: For each $n \ge 1$, let $X_i^{(n)}$, $1 \le i \le n$, be i.i.d. real-valued variables; define $S_0^{(n)} := 0$ and $S_j^{(n)} := \sum_{i=1}^{j} X_i^{(n)}$ for $1 \le j \le n$. Let (a_n) be positive numbers such that $a_n \to \infty$ and $\frac{a_n^2}{n} \to 0$, $n \to \infty$. Assume that there exists some $\eta > 0$ and a constant $\sigma^2 > 0$ such that, as $n \to \infty$,

$$\mathbf{E}(X_1^{(n)}) = o\Big(\frac{a_n}{n}\Big), \qquad \sup_{n \ge 1} \mathbf{E}(|X_1^{(n)}|^{2+\eta}) < \infty, \qquad \mathrm{Var}(X_1^{(n)}) \to \sigma^2. \tag{4.1.4}$$

The following estimate is essentially due to Mogulskii [74]:

Proposition 4.1.1 (A triangular version of Mogulskii [74]) *Assume (4.1.4). Let $g_1 < g_2$ be continuous functions on $[0, 1]$ with $g_1(0) < 0 < g_2(0)$. Consider the measurable event*

$$F_n := \Big\{ g_1(\frac{i}{n}) \le \frac{S_i^{(n)}}{a_n} \le g_2(\frac{i}{n}), \ \textit{for } 1 \le i \le n \Big\}.$$

We have

$$\lim_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}(F_n) = -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{\mathrm{d}t}{[g_2(t) - g_1(t)]^2}. \tag{4.1.5}$$

Moreover, for any $b > 0$,

$$\lim_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}\Big\{ F_n, \ \frac{S_n^{(n)}}{a_n} \ge g_2(1) - b \Big\} = -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{\mathrm{d}t}{[g_2(t) - g_1(t)]^2}. \tag{4.1.6}$$

If the law of $X_1^{(n)}$ does not depend on n (in which case we can even take $\eta = 0$), Proposition 4.1.1 is Mogulskii [74]'s theorem. For a detailed proof of Proposition 4.1.1, see [38].

A useful consequence of Proposition 4.1.1 is as follows. Again, if the law of $X_1^{(n)}$ does not depend on n, we only need $X_1^{(n)}$ to have a finite second moment in order to have (4.1.4).

Corollary 4.1.2 *Assume that (4.1.4) is satisfied with $a_n := n^{1/3}$.*

(i) *Let $f : [0, 1] \to (0, \infty)$ be a continuous function, and let (f_n) be a sequence of continuous functions converging uniformly to f on $[0, 1]$. Then for any $b > 0$, when $n \to \infty$,*

$$\sup_{0 \le u \le b\, n^{1/3}} \mathbf{P}\Big(u \ge S_i^{(n)} \ge u - n^{1/3} f_n(\frac{i}{n}), \ 1 \le i \le n \Big) = \mathrm{e}^{-\frac{\pi^2\sigma^2}{2}(1+o(1)) n^{1/3} \int_0^1 \frac{\mathrm{d}t}{f^2(t)}}.$$

(ii) *For any $b > a > 0$, we have, as $n \to \infty$,*

$$\sum_{j=1}^n \mathrm{e}^{-b(n-j)^{1/3}} \mathbf{P}\Big(a n^{1/3} \ge S_i^{(n)} > a n^{1/3} - b(n-i)^{1/3}, \ \forall\, 1 \le i \le j \Big) = \mathrm{e}^{-\min\{b, \frac{3\pi^2\sigma^2}{2b^2}\}(1+o(1)) n^{1/3}}.$$

Proof of Corollary 4.1.2. We first prove (ii). Let $\varepsilon > 0$. Define $k := \lfloor \frac{1}{\varepsilon} \rfloor$, $n_\ell := \ell \lfloor \varepsilon n \rfloor$ for $\ell = 0, \cdots, k-1$ and $n_k := n$. By (4.1.5), the sum in (ii) is, for all large n and some constant c,

$$\begin{aligned}
&\le \sum_{\ell=1}^k \mathrm{e}^{-b(n-n_\ell)^{1/3}} \mathbf{P}\Big(a n^{1/3} \ge S_i^{(n)} > a n^{1/3} - b(n-i)^{1/3}, \ \forall\, 1 \le i \le n_{\ell-1} \Big) \\
&\le \sum_{\ell=1}^k \mathrm{e}^{-b(n-n_\ell)^{1/3}} \, \mathrm{e}^{-(\frac{3\pi^2\sigma^2}{2b^2} - \varepsilon)(n^{1/3} - (n-n_{\ell-1})^{1/3})} \\
&\le \mathrm{e}^{-\min\{b, \frac{3\pi^2\sigma^2}{2b^2}\}(1-c\varepsilon) n^{1/3}}.
\end{aligned}$$

This proves the upper bound in (ii) as ε can be arbitrarily small. The lower bound is easier: we only need to consider two terms: $j = \lfloor \varepsilon n \rfloor$ and $j = n$, and apply again (4.1.5).

The proof of (i) goes along similar lines by cutting the interval $\{0 \le u \le b\, n^{1/3}\}$ into smaller intervals of length of order εn with small $\varepsilon > 0$, using monotonicity and applying Proposition 4.1.1. The details are omitted. $\square$

We now proceed to the proof of Theorem 4.0.7: if $\inf_{t\in[0,\,1]} \mathbf{E}\{\sum_{|x|=1} \mathrm{e}^{-tV(x)}\} = 1$ and $\mathbf{E}\{\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\} \le 0$, then on the set of non-extinction,

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \min_{|x|=n} \overline{V}(x) = \Big(\frac{3\pi^2\sigma_\theta^2}{2}\Big)^{1/3}, \qquad \mathbf{P}\text{-a.s.,}$$

where $\sigma_\theta^2 := \frac{1}{\theta}\mathbf{E}\{\sum_{|x|=1} V(x)^2\mathrm{e}^{-\theta V(x)}\}$ and $\theta \in (0,\,1]$ is such that $\mathbf{E}\{\sum_{|x|=1} V(x)\mathrm{e}^{-\theta V(x)}\} = 0$.

Without loss of generality, we can assume $\theta = 1$. Indeed, if $0 < \theta < 1$, then by considering $\widetilde{V}(x) := \theta V(x)$, we have $\inf_{t\in[0,\,1]} \mathbf{E}(\sum_{|x|=1} \mathrm{e}^{-t\widetilde{V}(x)}) = 1$ and $\mathbf{E}(\sum_{|x|=1} \widetilde{V}(x)\mathrm{e}^{-\widetilde{V}(x)}) = 0$, so that $\frac{1}{n^{1/3}} \min_{|x|=n} \max_{y\in\rrbracket\varnothing,\,x\rrbracket} \widetilde{V}(y) \to (\frac{3\pi^2\widetilde{\sigma}^2}{2})^{1/3}$ $\mathbf{P}$-almost surely on the set of non-extinction, where $\widetilde{\sigma}^2 := \mathbf{E}\{\sum_{|x|=1} \widetilde{V}(x)^2\mathrm{e}^{-\widetilde{V}(x)}\}$.

So we only need to prove Theorem 4.0.7 in the case $\theta = 1$. In the rest of the section, we assume $\mathbf{E}(\sum_{|x|=1} \mathrm{e}^{-V(x)}) = 1$ and $\mathbf{E}(\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}) = 0$, and prove that, on the set of non-extinction,

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \min_{|x|=n} \overline{V}(x) = \Big(\frac{3\pi^2\sigma^2}{2}\Big)^{1/3}, \qquad \mathbf{P}\text{-a.s.,} \tag{4.1.7}$$

with $\sigma^2 := \sigma_1^2 = \mathbf{E}\{\sum_{|x|=1} V(x)^2\mathrm{e}^{-V(x)}\}$. For the sake of clarity, we prove the upper and lower bounds in distinct parts.

Proof of (4.1.7): lower bound. We assume $\mathbf{E}(\sum_{|x|=1} \mathrm{e}^{-V(x)}) = 1$ and $\mathbf{E}(\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}) = 0$.

Let $0 < a < (\frac{3\pi^2\sigma^2}{2})^{1/3}$ and $b := (\frac{3\pi^2\sigma^2}{2})^{1/3}$. Let $n \ge 1$. For all $|x| = n$, let

$$H_x := \inf\{j \in [1, n] : V(x_j) \le an^{1/3} - b(n-j)^{1/3}\}, \qquad \inf\varnothing := \infty.$$

Assume there exists a vertex x with $|x| = n$ such that $\overline{V}(x) \le an^{1/3}$. Then $H_x \le n$; writing $j := H_x$ and $y := x_j$, we have, for all $i < j$, $an^{1/3} \ge V(y_i) > an^{1/3} - b(n-i)^{1/3}$ and $V(y) \le an^{1/3} - b(n-j)^{1/3}$. Therefore, by writing

$$U_j := \sum_{|y|=j} \mathbf{1}_{\{V(y)\le an^{1/3}-b(n-j)^{1/3},\ an^{1/3}\ge V(y_i)>an^{1/3}-b(n-i)^{1/3},\,\forall i<j\}},$$

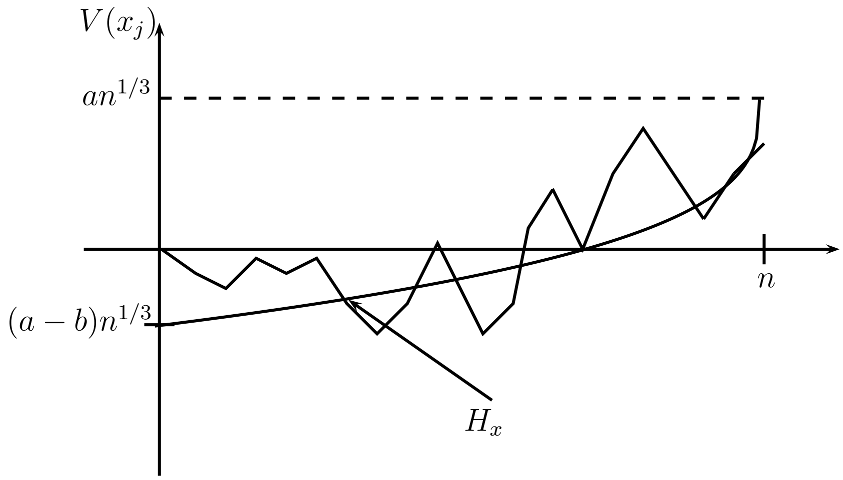


Figure 4.1: H_x

we obtain:

$$\mathbf{P}\Big(\min_{|x|=n} \overline{V}(x) \le an^{1/3}\Big) \le \mathbf{P}\Big(\bigcup_{j=1}^n \{U_j \ge 1\}\Big) \le \sum_{j=1}^n \mathbf{E}(U_j).$$

By (4.1.3), we have $\mathbf{E}(U_j) = \mathbf{E}[\mathrm{e}^{S_j} \mathbf{1}_{\{S_j \le an^{1/3} - b(n-j)^{1/3},\, an^{1/3} \ge S_i > an^{1/3} - b(n-i)^{1/3},\, \forall i<j\}}]$. Hence

$$\mathbf{P}\Big(\min_{|x|=n} \overline{V}(x) \le an^{1/3}\Big) \le \sum_{j=1}^n \mathrm{e}^{an^{1/3} - b(n-j)^{1/3}} \mathbf{P}\Big(an^{1/3} \ge S_i > an^{1/3} - b(n-i)^{1/3},\, \forall i < j\Big).$$

Applying Corollary 4.1.2 (ii) and noting that $\min\{b, \frac{3\pi^2\sigma^2}{2b^2}\} = (\frac{3\pi^2\sigma^2}{2})^{1/3}$, we get that, for any $0 < a < (\frac{3\pi^2\sigma^2}{2})^{1/3}$,

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\Big(\min_{|x|=n} \overline{V}(x) \le an^{1/3}\Big) \le a - \Big(\frac{3\pi^2\sigma^2}{2}\Big)^{1/3}, \tag{4.1.8}$$

which implies $\sum_n \mathbf{P}\{\min_{|x|=n} \overline{V}(x) \le an^{1/3}\} < \infty$. The lower bound in (4.1.7) follows from the Borel–Cantelli lemma, as a can be as close to $(\frac{3\pi^2\sigma^2}{2})^{1/3}$ as possible. □

Proof of (4.1.7): upper bound. Assume $\mathbf{E}\{\sum_{|x|=1} \mathrm{e}^{-V(x)}\} = 1$ and $\mathbf{E}\{\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\} = 0$.

Let $n \ge 1$ and $b > a > \varepsilon > 0$. The key step in the proof of the upper bound in (4.1.7) is the following estimate, which is a consequence of the Paley–Zygmund inequality (see [38] for a proof): For any Borel sets $I_{i,n} \subset \mathbb{R}$, $1 \le i \le n$, and any integer $r_n \ge 1$, we have

$$\mathbf{P}\Big\{\exists |x| = n:\ V(x_i) \in I_{i,n}\,,\ \forall 1 \le i \le n\Big\} \ge \frac{\mathbf{E}[\mathrm{e}^{S_n}\, \mathbf{1}_{\{S_i \in I_{i,n}\,,\ \nu_{i-1} \le r_n,\ \forall\, 1 \le i \le n\}}]}{1 + (r_n - 1)\sum_{j=1}^n h_{j,n}}, \tag{4.1.9}$$

where

$$h_{j,n} := \sup_{u\in I_{j,n}} \mathbf{E}\Big(\mathrm{e}^{S_{n-j}}\mathbf{1}_{\{S_i\in I_{i+j,n}-u,\, \forall 0\le i\le n-j\}}\Big),$$

and $I_{i+j,n} - u := \{v-u: \ v\in I_{i+j,n}\}$. [We recall that $(S_i - S_{i-1}, \nu_{i-1})$, $i\ge 1$, are i.i.d. random vectors (with $S_0 := 0$) whose common distribution is given by (4.1.2).]

We choose $r_n := \lfloor \mathrm{e}^{n^{1/4}}\rfloor$ and $I_{i,n} := [(a-\varepsilon)n^{1/3} - b(n-i)^{1/3},\, an^{1/3}]$. In particular, $\{\exists |x| = n: \ V(x_i)\in I_{i,n}\,,\ \forall 1\le i\le n\} \subset \{\min_{|x|=n}\overline{V}(x)\le an^{1/3}\}$. It follows from (4.1.9) that

$$\mathbf{P}\Big(\min_{|x|=n}\overline{V}(x)\le an^{1/3}\Big) \ge \frac{\mathrm{e}^{(a-\varepsilon)n^{1/3}}\,\mathbf{P}\{S_i\in I_{i,n}\,,\ \nu_{i-1}\le \mathrm{e}^{n^{1/4}},\ \forall\, 1\le i\le n\}}{1+\mathrm{e}^{n^{1/4}}\sum_{j=1}^n h_{j,n}}.$$

Let $X_j^{(n)}$, $1\le j\le n$, be i.i.d. random variables such that $X_1^{(n)}$ has the same distribution as S_1 conditioned on $\{\nu_0\le \mathrm{e}^{n^{1/4}}\}$. Let $S_0^{(n)} := 0$ and $S_j^{(n)} := X_1^{(n)} + ... + X_j^{(n)}$ for $1\le j\le n$. Then

$$\begin{aligned}
&\mathbf{P}\{S_i\in I_{i,n}\,,\ \nu_{i-1}\le \mathrm{e}^{n^{1/4}},\ \forall\, 1\le i\le n\}\\
=\ &[\,\mathbf{P}(\nu_0\le \mathrm{e}^{n^{1/4}})]^n\,\mathbf{P}\Big\{\max_{0\le k\le n} S_k^{(n)}\le an^{1/3},\ S_i^{(n)}\ge (a-\varepsilon)n^{1/3} - b(n-i)^{1/3},\ \forall 1\le i\le n\Big\}.
\end{aligned}$$

The second probability expression on the right-hand side is, according to Proposition 4.1.1 (we easily check that condition (4.1.4) is satisfied), $= \exp\{-(1+o(1))\frac{\pi^2\sigma^2}{2}n^{1/3}\int_0^1 \frac{\mathrm{d}t}{(\varepsilon+b(1-t)^{1/3})^2}\}$, which is bounded by $\exp\{-(\frac{3\pi^2\sigma^2}{2b^2} - c_1(\varepsilon))n^{1/3}\}$ for all sufficiently large n, with $c_1(\varepsilon)$ denoting a constant such that $\lim_{\varepsilon\to 0} c_1(\varepsilon) = 0$. On the other hand, for sufficiently small $\eta>0$, $\mathbf{E}[(\nu_0)^\eta] = \mathbf{E}(N^\eta\sum_{i=1}^N A_i) < \infty$ by Hölder's inequality and (2.5.1); thus $[\mathbf{P}(\nu_0\le \mathrm{e}^{n^{1/4}})]^n \to 1$ as $n\to\infty$. Accordingly, for all sufficiently large n and some constant $c_2(\varepsilon)$ satisfying $\lim_{\varepsilon\to 0} c_2(\varepsilon) = 0$,

$$\mathbf{P}\Big(\min_{|x|=n}\overline{V}(x)\le an^{1/3}\Big) \ge \frac{\exp\{n^{1/3}[a-\frac{3\pi^2\sigma^2}{2b^2} - c_2(\varepsilon)]\}}{1+\mathrm{e}^{n^{1/4}}\sum_{j=1}^n h_{j,n}}. \tag{4.1.10}$$

We now estimate $\sum_{j=1}^n h_{j,n}$. By definition,

$$\begin{aligned}
h_{j,n} &= \sup_{0\le u\le \varepsilon n^{1/3}+b(n-j)^{1/3}} \mathbf{E}\Big(\mathrm{e}^{S_{n-j}}\mathbf{1}_{\{u\ge S_i\ge u-\varepsilon n^{1/3}-b(n-j-i)^{1/3},\,\forall\, i\le n-j\}}\Big)\\
&\le \sup_{0\le u\le \varepsilon n^{1/3}+b(n-j)^{1/3}} \mathrm{e}^u\,\mathbf{P}\Big(u\ge S_i\ge u-\varepsilon n^{1/3} - b(n-j-i)^{1/3},\ \forall\, i\le n-j\Big).
\end{aligned}$$

Let A be an integer such that $A\ge \frac{1}{\varepsilon^2}$. Let $n_\ell := \ell\lfloor\frac{n}{A}\rfloor$ for $\ell = 0, 1, \cdots, A-1$ and $n_A := n$. If $j\in [n_\ell,\, n_{\ell+1}]\cap\mathbb{Z}$ (for some $0\le \ell\le A-1$), then

$$h_{j,n}\le \mathrm{e}^{\varepsilon n^{1/3}+b(n-n_\ell)^{1/3}} \sup_{0\le u\le (b+\varepsilon)n^{1/3}} \mathbf{P}\Big(u\ge S_i\ge u-\varepsilon n^{1/3} - b(n-n_\ell - i)^{1/3},\ \forall i\le n-n_{\ell+1}\Big).$$

We now bound the supremum on the right-hand side. If ℓ is such that $1-\frac{\ell+1}{A}\le\varepsilon$, then we simply say that the supremum is bounded by 1, so that $\max_{n_\ell\le j\le n_{\ell+1}} h_{j,n}\le \mathrm{e}^{\varepsilon n^{1/3}+b(n-n_\ell)^{1/3}}$. If $1-\frac{\ell+1}{A}>\varepsilon$, we bound the supremum by applying Corollary 4.1.2 (i) to $f(t):=\frac{\varepsilon}{(1-\frac{\ell+1}{A})^{1/3}}+b(\frac{A-\ell}{A-(\ell+1)}-t)^{1/3}$: since $f(t)\le\varepsilon^{2/3}+b(1+\frac{1}{\varepsilon A}-t)^{1/3}\le\varepsilon^{2/3}+b(1+\varepsilon-t)^{1/3}$ (using $A\ge\frac{1}{\varepsilon^2}$ for the second inequality), we have $\int_0^1\frac{\mathrm{d}t}{f^2(t)}\ge\frac{3}{b^2}-c_3(\varepsilon)$, with $c_3(\varepsilon)$ denoting a constant satisfying $\lim_{\varepsilon\to0}c_3(\varepsilon)=0$; hence by Corollary 4.1.2 (i),

$$\max_{n_\ell\le j\le n_{\ell+1}} h_{j,n}\le \mathrm{e}^{\varepsilon n^{1/3}+b(n-n_\ell)^{1/3}-(\frac{3\pi^2\sigma^2}{2b^2}-c_3(\varepsilon))(n-n_{\ell+1})^2}.$$

Consequently, for all sufficiently large n and a constant $c(\varepsilon)$ satisfying $\lim_{\varepsilon\to0}c(\varepsilon)=0$,

$$\max_{0\le j\le n} h_{j,n}=\max_{0\le\ell\le A-1}\max_{n_\ell\le j\le n_{\ell+1}} h_{j,n}\le \mathrm{e}^{n^{1/3}[(b-\frac{3\pi^2\sigma^2}{2b^2})^++c(\varepsilon)]},$$

where $u^+:=\max\{u,\,0\}$. In view of (4.1.10), we obtain that, for any $b>a>0$,

$$\liminf_{n\to\infty}\frac{1}{n^{1/3}}\log\mathbf{P}\Big(\min_{|x|=n}\overline{V}(x)\le an^{1/3}\Big)\ge-\big(b-\frac{3\pi^2\sigma^2}{2b^2}\big)^++a-\frac{3\pi^2\sigma^2}{2b^2}. \tag{4.1.11}$$

We now fix $a>(\frac{3\pi^2\sigma^2}{2})^{1/3}$ and $\eta>0$. We can choose $b>a$ sufficiently close to a such that $(b-\frac{3\pi^2\sigma^2}{2b^2})^+-a+\frac{3\pi^2\sigma^2}{2b^2}<\eta$; accordingly, for all sufficiently large n,

$$\mathbf{P}\Big(\min_{|x|=n}\overline{V}(x)\le an^{1/3}\Big)\ge \mathrm{e}^{-\eta\, n^{1/3}}. \tag{4.1.12}$$

From here, it is routine (McDiarmid [72]) to obtain the upper bound in (4.1.7); we produce the details for the sake of completeness. Let $R_n:=\inf\{k:\ \#\{x:\,|x|=k\}\ge \mathrm{e}^{2\eta\, n^{1/3}}\}$. For all large n,

$$\begin{aligned}
&\mathbf{P}\Big\{R_n<\infty,\ \max_{k\in[\frac{n}{2},\,n]}\min_{|x|=k+R_n}\overline{V}(x)>\max_{|y|=R_n}\overline{V}(y)+an^{1/3}\Big\}\\
\le\ &\sum_{k\in[\frac{n}{2},\,n]}\mathbf{P}\Big\{R_n<\infty,\ \min_{|x|=k+R_n}\overline{V}(x)>\max_{|y|=R_n}\overline{V}(y)+an^{1/3}\Big\}\\
\le\ &\sum_{k\in[\frac{n}{2},\,n]}\Big[\mathbf{P}\Big\{\min_{|x|=k}\overline{V}(x)>an^{1/3}\Big\}\Big]^{\lfloor \mathrm{e}^{2\eta n^{1/3}}\rfloor},
\end{aligned}$$

which, according to (4.1.12), is summable in n. By the Borel–Cantelli lemma, $\mathbf{P}$-a.s. for all large n, we have either $R_n=\infty$, or $\max_{k\in[\frac{n}{2},\,n]}\min_{|x|=k+R_n}\overline{V}(x)\le\max_{|y|=R_n}\overline{V}(y)+an^{1/3}$.

By the law of large numbers for the branching random walk (Biggins [7]), there exists a constant $c\in(0,\,\infty)$ such that $\frac{1}{n}\max_{|y|=n}V(y)\to c$, $\mathbf{P}$-almost surely upon the system's survival. In particular, upon survival, $\max_{|y|=n} V(y) \le 2cn$, $\mathbf{P}$-almost surely for all large n. Consequently, upon the system's survival, $\mathbf{P}$-almost surely for all large n, we have either $R_n = \infty$, or $\max_{k\in[\frac{n}{2},n]} \min_{|x|=k+R_n} \overline{V}(x) \le 2cR_n + an^{1/3}$.

Recall that the number of particles in each generation forms a supercritical Galton–Watson tree. In particular, conditionally on the system's survival, $\frac{\#\{u:\, |u|=k\}}{(\mathbf{E}N)^k}$ converges a.s. to a (strictly) positive random variable when $k\to\infty$, which implies $R_n \sim 2\eta \frac{n^{1/3}}{\log(\mathbf{E}N)}$ $\mathbf{P}$-a.s. $(n\to\infty)$, and $\max_{k\in[\frac{n}{2},n]} \min_{|x|=k+R_n} \overline{V}(x) \ge \min_{|x|=n} \overline{V}(x)$ $\mathbf{P}$-almost surely for all large n. As a consequence, upon the system's survival, we have, $\mathbf{P}$-almost surely for all large n,

$$\min_{|x|=n} \overline{V}(x) \le \frac{5c\eta}{\log(\mathbf{E}N)}\, n^{1/3} + an^{1/3}.$$

Since a (resp. η) can be as close to $(\frac{3\pi^2\sigma^2}{2})^{1/3}$ (resp. 0) as possible, this yields the upper bound in (4.1.7), and completes the proof of Theorem 4.0.7. □

Our proof of Theorem 4.0.7 gives the following deviation probability of $\min_{|x|=n} \overline{V}(x)$, which may be of independent interest.

Proposition 4.1.3 *Assume $\psi(1) = \psi'(1) = 0$. For any $0 < a \le (\frac{3\pi^2\sigma^2}{2})^{1/3}$, we have*

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\Big(\min_{|x|=n} \overline{V}(x) \le an^{1/3} \Big) = a - \Big(\frac{3\pi^2\sigma^2}{2}\Big)^{1/3}. \tag{4.1.13}$$

Proof. If $0 < a < (\frac{3\pi^2\sigma^2}{2})^{1/3}$, the upper and lower bounds in (4.1.13) follow from (4.1.8) and (4.1.11), respectively, whereas if $a = (\frac{3\pi^2\sigma^2}{2})^{1/3}$, only the lower bound in (4.1.13) needs proved, which follows immediately from (4.1.11). □

Remark 4.1.4 Assume $\psi(1) = \psi'(1) = 0$. Theorem 4.0.7 says that, on the set of non-extinction, $\mathbf{P}$-almost surely for $n\to\infty$, there exists x_n with $|x_n| = n$ such that $\overline{V}(x_n) = (1+o(1))(\frac{3\pi^2\sigma^2}{2})^{1/3} n^{1/3}$. One may wonder whether the vertices (x_n) can be chosen to form an infinite ray (i.e., each x_n is a child of x_{n-1}). The answer is no: Jaffuel [48] proves that this is possible only if we enlarge the function $(\frac{3\pi^2\sigma^2}{2})^{1/3} n^{1/3}$ to $(\frac{81\pi^2\sigma^2}{8})^{1/3} n^{1/3}$. □

4.2 An estimate for one-dimensional random walks

We present in this section a probability estimate for one-dimensional random walks. It will be used in the proofs of Theorems 2.5.4 and 2.5.5 in the forthcoming sections. For each

$n \geq 1$, let $X_i^{(n)}$, $1 \leq i \leq n$, be i.i.d. real-valued variables; let $S_0^{(n)} := 0$ and $S_j^{(n)} := \sum_{i=1}^j X_i^{(n)}$ for $1 \leq j \leq n$. Let (a_n) be positive numbers such that $a_n \to \infty$ and $\frac{a_n^2}{n} \to 0$, $n \to \infty$. We write $\overline{S}_j^{(n)} := \max_{1\leq i\leq j} S_i^{(n)}$ for $1 \leq j \leq n$.

Proposition 4.2.1 *Assume (4.1.4). Let $f : [0,1] \to (0,\infty)$ be a continuous function. For $\delta \geq 0$, we consider the event*

$$G_\delta(n) := \Big\{(1+\delta)\overline{S}_j^{(n)} - S_j^{(n)} \leq a_n\, f(\frac{j}{n}),\ \forall 1 \leq j \leq n\Big\}.$$

(i) *If $\delta = 0$, then*

$$\lim_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}\Big\{G_0(n)\Big\} = -\frac{\pi^2\sigma^2}{8}\int_0^1 \frac{\mathrm{d}s}{f^2(s)}.$$

Moreover, for any fixed $0 < b < 1$,

$$\lim_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}\Big\{G_0(n),\, \overline{S}_n^{(n)} - S_n^{(n)} \leq b\, a_n\, f(1)\Big\} = -\frac{\pi^2\sigma^2}{8}\int_0^1 \frac{\mathrm{d}s}{f^2(s)}.$$

(ii) *If $\delta > 0$, then*

$$\lim_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}\Big\{G_\delta(n)\Big\} = -\frac{\pi^2\sigma^2}{2(1+\delta)^2}\int_0^1 \frac{\mathrm{d}s}{f^2(s)}.$$

We mention that for the centered random walk (S_n) given in (4.1.3), assumption (4.1.4) is obviously satisfied. Hence Proposition 4.1.1 as well as Corollary 4.2.2 below, hold also for (S_n).

Proof of Proposition 4.2.1. (i) Let $0 < \varepsilon < \frac{1}{4}\min\{b,\, \min_{0\leq t\leq 1} f(t)\}$ and let A be a large integer. Consider a sufficiently large n such that $\sup_{0\leq s<t\leq 1,\, t-s\leq 2Aa_n^2/n} |f(t) - f(s)| \leq \varepsilon$. Let $m = \lfloor \frac{n}{A^2 a_n^2} \rfloor$. For $0 \leq k < Am$, let $r_k := k\lfloor Aa_n^2 \rfloor$ and $r_{Am} := n$. Note that $\lfloor Aa_n^2 \rfloor \leq r_{Am} - r_{Am-1} \leq 2\lfloor Aa_n^2 \rfloor$. Let $\ell = 0, 1, ..., A-1$ and $k \in [\ell m,\, (\ell+1)m - 1] \cap \mathbb{Z}$. For all $r_k \leq j < r_{k+1}$, $|f(\frac{j}{n}) - f(\frac{\ell}{A})| \leq \varepsilon$. Define

$$E_n^{(\pm)} := \bigcap_{\ell=0}^{A-1} \bigcap_{k=\ell m}^{(\ell+1)m-1} \Big\{\overline{S}_j^{(n)} - S_j^{(n)} \leq a_n\, (f(\frac{\ell}{A}) \pm \varepsilon),\ \forall\ r_k \leq j < r_{k+1}\Big\}.$$

Then

$$\begin{aligned} \mathbf{P}\Big(G_0(n)\Big) &\leq \mathbf{P}\left(E_n^{(+)}\right), \\ \mathbf{P}\Big(G_0(n), \overline{S}_n^{(n)} - S_n^{(n)} \leq b\, a_n\, f(\frac{j}{n})\Big) &\geq \mathbf{P}\Big(E_n^{(-)} \cap \bigcap_{0\leq k\leq Am} \{\overline{S}_{r_k}^{(n)} - S_{r_k}^{(n)} \leq \varepsilon a_n\}\Big). \end{aligned}$$

Observe that for any r_k, conditionally on $\sigma\{S_j^{(n)}, 0 \le j \le r_k\}$ and on $\{\overline{S}_{r_k}^{(n)} - S_{r_k}^{(n)} = x\}$, the reflecting process $(\overline{S}_{i+r_k}^{(n)} - S_{i+r_k}^{(n)},\ 0 \le i \le r_{k+1} - r_k)$ has the same law as $(\max\{x, \overline{S}_i^{(n)}\} - S_i^{(n)},\ 0 \le i \le r_{k+1} - r_k)$. Using this observation for all k, we see that

$$\mathbf{P}(E_n^{(+)}) \le \prod_{\ell=0}^{A-1} \prod_{k=\ell m}^{(\ell+1)m-1} \mathbf{P}\Big\{ \max_{0\le i<r_{k+1}-r_k} (\overline{S}_i^{(n)} - S_i^{(n)}) \le a_n\, (f(\frac{\ell}{A}) + \varepsilon)\Big\}, \tag{4.2.1}$$

$$\mathbf{P}\Big(E_n^{(-)} \cap \bigcap_{0\le k\le Am} \{\overline{S}_{r_k}^{(n)} - S_{r_k}^{(n)} \le \varepsilon a_n\}\Big) \ge \prod_{\ell=0}^{A-1} \prod_{k=\ell m}^{(\ell+1)m-1} \mathbf{P}\Big\{\Upsilon_k\Big\}, \tag{4.2.2}$$

with

$$\Upsilon_k := \Big\{ \max_{0\le i<r_{k+1}-r_k} (\overline{S}_i^{(n)} - S_i^{(n)}) \le a_n\, (f(\frac{\ell}{A}) - 2\varepsilon), \overline{S}_{r_{k+1}-r_k}^{(n)} - S_{r_{k+1}-r_k}^{(n)} < \varepsilon a_n,\ \overline{S}_{r_{k+1}-r_k}^{(n)} > \varepsilon a_n \Big\}.$$

Now, we prove the upper bound in (i). By (4.2.1),

$$\frac{a_n^2}{n} \log \mathbf{P}(E_n^{(+)}) \le \frac{m a_n^2}{n} \sum_{\ell=0}^{A-1} \log \mathbf{P}\Big\{ \overline{S}_i^{(n)} - S_i^{(n)} \le a_n\, (f(\frac{\ell}{A}) + \varepsilon),\ \forall\, 0 \le i < \lfloor A a_n^2 \rfloor \Big\}.$$

According to Donsker's invariance principle,[1] the probability term on the right-hand side converges, when $n \to \infty$, to

$$\mathbf{P}\Big\{ \sup_{0\le t\le 1} (\overline{W}(t) - W(t)) \le \frac{1}{\sigma\sqrt{A}} (f(\frac{\ell}{A}) + \varepsilon) \Big\},$$

where W is a standard one-dimensional Brownian motion, and $\overline{W}(t) = \sup_{0\le s\le t} W(s)$. By Lévy's identity, $(\overline{W}(t) - W(t),\ t \ge 0)$ is distributed as $(|W(t)|,\ t \ge 0)$; thus we have

$$\mathbf{P}\Big\{ \sup_{0\le t\le 1} (\overline{W}(t) - W(t)) \le u \Big\} = \mathrm{e}^{-(1+o(1))\frac{\pi^2}{8u^2}}, \qquad u \to 0, \tag{4.2.3}$$

see Chung [18]. As a consequence, for all sufficiently large A, say $A \ge A_0 = A_0(\varepsilon, \sigma, f)$,

$$\log \mathbf{P}\Big\{ \sup_{0\le t\le 1} (\overline{W}(t) - W(t)) \le \frac{1}{\sigma\sqrt{A}} (f(\frac{\ell}{A}) + \varepsilon) \Big\} \le -\frac{(1-\varepsilon)\pi^2 \sigma^2 A}{8(f(\frac{\ell}{A}) + \varepsilon)^2}.$$

Since $m \sim \frac{n}{a_n^2 A^2}$, we get, for $A \ge A_0$,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{a_n^2}{n} \log \mathbf{P}(E_n^{(+)}) &\le \frac{1}{A^2} \sum_{\ell=0}^{A-1} \log \mathbf{P}\Big\{ \sup_{0\le t\le 1} (\overline{W}(t) - W(t)) \le \frac{1}{\sigma\sqrt{A}} (f(\frac{\ell}{A}) + \varepsilon) \Big\} \\
&\le -\frac{\pi^2\sigma^2}{8} \frac{1-\varepsilon}{A} \sum_{\ell=0}^{A-1} \frac{1}{(f(\frac{\ell}{A}) + \varepsilon)^2}.
\end{aligned}$$

[1] Finite-dimensional convergence is checked by Lindeberg's condition in the central limit theorem, whereas tightness is proved via a standard argument as in Billingsley [11].

Letting $A\to\infty$ and then $\varepsilon\to 0$, we get the upper bound in (i):

$$\limsup_{n\to\infty}\frac{a_n^2}{n}\,\log\mathbf{P}\Big\{\overline{S}_i^{(n)}-S_j^{(n)}\le a_n\,f(\frac{j}{n}),\ \forall 1\le j\le n\Big\}\le-\frac{\pi^2\sigma^2}{8}\int_0^1\frac{\mathrm{d}s}{f^2(s)}.$$

To prove the lower bound in (i), we go back to the events Υ_k in (4.2.2). Observe that coordinately for each $1\le i\le r_{k+1}-r_k$, all the three events in Υ_k are non-decreasing on $S_i^{(n)}-S_{i-1}^{(n)}$. By the FKG inequality,

$$\begin{aligned}\mathbf{P}\Big(\Upsilon_k\Big) &\ \ge\ \mathbf{P}\Big(\max_{0\le i<r_{k+1}-r_k}(\overline{S}_i^{(n)}-S_i^{(n)})\le a_n\,(f(\frac{\ell}{A})-2\varepsilon)\Big)\\ &\qquad\times\mathbf{P}\Big(\overline{S}_{r_{k+1}-r_k}^{(n)}-S_{r_{k+1}-r_k}^{(n)}<\varepsilon a_n\Big)\,\mathbf{P}\Big(\overline{S}_{r_{k+1}-r_k}^{(n)}>\varepsilon a_n\Big).\end{aligned}$$

Recall that $r_{k+1}-r_k=\lfloor Aa_n^2\rfloor$ for $0\le k<Am-1$, and $\lfloor Aa_n^2\rfloor\le r_{Am}-r_{Am-1}\le 2\lfloor Aa_n^2\rfloor$. Using Donsker's invariance principle again, we see that there exists a constant $c(\varepsilon)>0$ such that for all k, $\mathbf{P}(\overline{S}_{r_{k+1}-r_k}^{(n)}-S_{r_{k+1}-r_k}^{(n)}<\varepsilon a_n)\,\mathbf{P}(\overline{S}_{r_{k+1}-r_k}^{(n)}>\varepsilon a_n)\ge c(\varepsilon)$. From this, the lower bound in (i) follows in the same way as the upper bound in (i).

(ii) The proof of (ii) goes along the same lines as that of (i), except that instead of (4.2.3), we use the following estimate: for any $\delta>0$,

$$\mathbf{P}\Big\{\sup_{0\le s\le 1}((1+\delta)\overline{W}(s)-W(s))\le u\Big\}=\mathrm{e}^{-\frac{\pi^2}{2u^2}(1+o(1))},\qquad u\to 0. \tag{4.2.4}$$

To see why (4.2.4) holds, we denote by $L(t)$ the local time at 0 of W up to time t, and recall from Borodin and Salminen ([15], page 259, Formula 1.16.2) that, for $\lambda>0$,

$$\int_0^\infty \mathrm{e}^{-\lambda t}\,\mathbf{P}\Big(\sup_{s\le t}|W(s)|\le 1,\ L(t)\le\frac{1}{\delta}\Big)\,\mathrm{d}t=\frac{1}{\lambda}\Big(1-\frac{1}{\cosh(\sqrt{2\lambda})}\Big)\Big(1-\mathrm{e}^{-\frac{1}{\delta}\sqrt{\frac{\lambda}{2}}\coth(\sqrt{2\lambda})}\Big).$$

By analytic continuation, we get that for $0<\lambda<\frac{\pi^2}{2}$,

$$\int_0^\infty \mathrm{e}^{\lambda t}\,\mathbf{P}\Big(\sup_{s\le t}|W(s)|\le 1,\ L(t)\le\frac{1}{\delta}\Big)\,\mathrm{d}t=\frac{1}{\lambda}\Big(\frac{1}{\cos(\sqrt{2\lambda})}-1\Big)\Big(1-\mathrm{e}^{-\frac{1}{\delta}\sqrt{\frac{\lambda}{2}}\mathrm{cotan}(\sqrt{2\lambda})}\Big).$$

This implies, by means of a Tauberian theorem (see, for example, Theorem 3.2 of [40]), that

$$\mathbf{P}\Big(\sup_{0\le s\le t}|W(s)|\le 1,\ L(t)\le\frac{1}{\delta}\Big)=\mathrm{e}^{-(\frac{\pi^2}{2}+o(1))t},\qquad t\to\infty,$$

which, by scaling, is equivalent to $\mathbf{P}(\sup_{0\le s\le 1}|W(s)|\le u,\ L(1)\le\frac{u}{\delta})=\mathrm{e}^{-(\frac{\pi^2}{2u}+o(1))}$, $u\to 0$. For any $0<\varepsilon<1$, we have

$$\begin{aligned}\mathbf{P}\Big(\sup_{s\le 1}|W(s)|\le(1-\varepsilon)u,\ L(1)\le\frac{\varepsilon u}{\delta}\Big) &\ \le\ \mathbf{P}\Big(\sup_{s\le 1}(|W(s)|+\delta L(s))\le u\Big)\\ &\ \le\ \mathbf{P}\Big(\sup_{s\le 1}|W(s)|\le u,\ L(1)\le\frac{u}{\delta}\Big);\end{aligned}$$

therefore,

$$\mathbf{P}\Big\{\sup_{0\le s\le 1}(|W(s)|+\delta L(s))\le u\Big\}=\mathrm{e}^{-\frac{\pi^2}{2u^2}(1+o(1))},\qquad u\to 0.$$

By Lévy's identity, the two processes $(\overline{W}-W,\,\overline{W})$ and $(|W|,\,L)$ have the same law; consequently, this implies (4.2.4). $\square$

The following corollary follows from Proposition 4.2.1 exactly as Corollary 4.1.2 follows from Proposition 4.1.1.

Corollary 4.2.2 *Assume that (4.1.4) is satisfied with $a_n = n^{1/3}$. Let $a>0$ and $\delta>0$. Then for $n\to\infty$,*

$$\sum_{j=1}^n \mathrm{e}^{-a(n-j)^{1/3}}\mathbf{P}\Big(\overline{S}_i^{(n)}-S_i^{(n)}\le a(n-i)^{1/3},\,\forall\, 1\le i\le j\Big) = \mathrm{e}^{-\min\{a,\,\frac{3\pi^2\sigma^2}{8a^2}\}(1+o(1))n^{1/3}},$$

$$\sum_{j=1}^n \mathrm{e}^{-a(n-j)^{1/3}}\mathbf{P}\Big((1+\delta)\overline{S}_i^{(n)}-S_i^{(n)}\le a(n-i)^{1/3},\,\forall\, 1\le i\le j\Big) = \mathrm{e}^{-\min\{a,\,\frac{3\pi^2\sigma^2}{2(1+\delta)^2a^2}\}(1+o(1))n^{1/3}}.$$

4.3 Proof of Theorem 2.5.5

We assume $\inf_{t\in[0,\,1]}\psi(t)=0$ and $\psi'(1)\ge 0$ in this section. Let $\theta\in(0,\,1]$ be such that $\psi'(\theta)=0$ as in (2.5.4). By Theorem 4.0.7 and (4.0.2), we get that, on the set of non-extinction,

$$\liminf_{n\to\infty}\frac{1}{n^{1/3}}\log\varrho_n\ge -\alpha_\theta^{1/3},\qquad \mathbf{P}\text{-a.s.},$$

where $\alpha_\theta := \frac{3\pi^2}{2\theta}\mathbf{E}[\sum_{i=1}^N A_i^\theta(\log A_i)^2] = \frac{3\pi^2}{2\theta}\mathbf{E}[\sum_{|x|=1}V(x)^2\mathrm{e}^{-\theta V(x)}]$, and $\varrho_n := P_\omega\{\tau_n<\tau_0\}$ is as in (4.0.2). In view of Fact 4.0.6, it remains only to check that if $\psi'(1)>0$ (i.e., if $\theta<1$), then we have, on the set of non-extinction,

$$\limsup_{n\to\infty}\frac{1}{n^{1/3}}\log\varrho_n\le -\alpha_\theta^{1/3},\qquad \mathbf{P}\text{-a.s.} \tag{4.3.1}$$

We do not assume $\psi'(1)>0$ for the moment (so θ can be 1, and the inequality (4.3.2) below can also be used in the proof of Theorem 2.5.4 in the next section). Let $a>0$, $n\ge 1$ and $\delta\ge 0$. For any y with $|y|\le n$, say $|y|=j$, we introduce the following event:

$$E_\delta(y)=\Big\{(1+\delta)\overline{V}(y)-V(y)\ge\frac{a}{\theta}(n-j)^{1/3}\Big\}\cap\bigcap_{i=1}^{j-1}\Big\{(1+\delta)\overline{V}(y_i)-V(y_i)<\frac{a}{\theta}(n-i)^{1/3}\Big\},$$

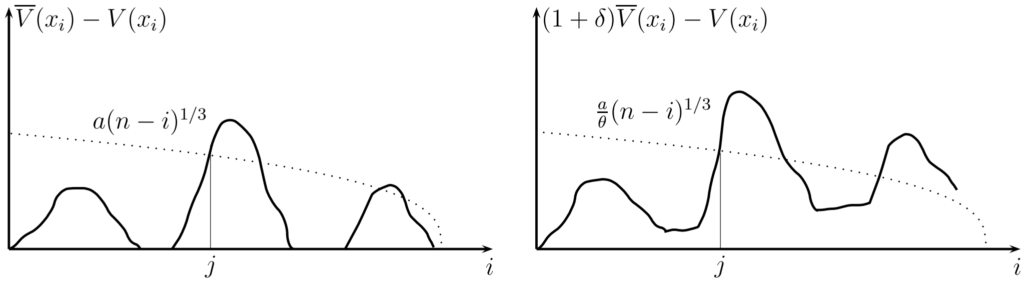


Figure 4.2: $\inf\{i; E_\delta(x_i)$ holds.$\}$

where y_i is the unique vertex of $[\![\varnothing, y]\!]$ that is in the i-th generation, whereas $\overline{V}(x) := \max_{z\in]\!]\varnothing, x]\!]} V(z)$ as in (4.0.3).

Let as before $\tau_n := \inf\{i \geq 1 : |X_i| = n\}$ and $T(x) := \inf\{k \geq 0 : X_k = x\}$. Consider any vertex x with $|x| = n$. Let $j = j(x) \in [1, n] \cap \mathbb{Z}$ be the smallest integer such that $(1+\delta)\overline{V}(x_j) - V(x_j) \geq \frac{a}{\theta}(n-j)^{1/3}$. Such a j exists. Moreover, we have $T(x) \geq T(x_j)$, and $E_\delta(x_j)$ holds. Consequently,

$$\tau_n = \inf_{|x|=n} T(x) \geq \min_{1\leq j\leq n} \inf\{T(y) : |y| = j \text{ and } E_\delta(y) \text{ holds}\},$$

so that $\varrho_n = P_\omega\{\tau_n < \tau_0\} \leq \sum_{j=1}^n \sum_{|y|=j} \mathbf{1}_{E_\delta(y)} P_\omega\{T(y) < \tau_0\}$. By (4.0.5), we obtain:

$$\varrho_n \leq \sum_{j=1}^n \sum_{|y|=j} \mathbf{1}_{E_\delta(y)}\, \omega(\varnothing, y_1)\, \mathrm{e}^{V(y_1)-\overline{V}(y)} = \omega(\varnothing, \overleftarrow{\varnothing}) \sum_{j=1}^n \sum_{|y|=j} \mathbf{1}_{E_\delta(y)}\, \mathrm{e}^{-\overline{V}(y)}, \tag{4.3.2}$$

which is bounded by $\sum_{j=1}^n \sum_{|y|=j} \mathbf{1}_{E_\delta(y)}\, \mathrm{e}^{-\overline{V}(y)}$.

We now assume furthermore $\psi'(1) > 0$, so that $\theta < 1$. We choose $\delta \in (0, \frac{1}{\theta} - 1)$. Since $(1+\delta)\theta < 1$, we have

$$\varrho_n^{(1+\delta)\theta} \leq \sum_{j=1}^n \sum_{|y|=j} \mathbf{1}_{E_\delta(y)}\, \mathrm{e}^{-(1+\delta)\theta\overline{V}(y)}.$$

Consider the branching random walk $\widetilde{V}(x) := \theta V(x)$ for any x. If we define $\widetilde{\psi}(t) := \log \mathbf{E}[\sum_{|x|=1} \mathrm{e}^{-t\widetilde{V}(x)}]$, then $\widetilde{\psi}(1) = \widetilde{\psi}'(1) = 0$. We apply formula (4.1.3) to $(\widetilde{V}(x))$, and obtain a centered one-dimensional random walk $(\widetilde{S}_i, 0 \leq i \leq n)$ with $\widetilde{\sigma}^2 := \mathbf{E}(\widetilde{S}_1^2) = \mathbf{E}[\sum_{|x|=1} \theta^2 V(x)^2 \mathrm{e}^{-\theta V(x)}]$ such that for $1 \leq j \leq n$ (with $\overline{\widetilde{S}}_i := \max_{1\leq k\leq i} \widetilde{S}_k$),

$$\begin{aligned}
\mathbf{E}\Big(\sum_{|y|=j} \mathbf{1}_{E_\delta(y)}\, \mathrm{e}^{-(1+\delta)\theta\overline{V}(y)}\Big) &= \mathbf{E}\Big(\mathrm{e}^{\widetilde{S}_j-(1+\delta)\overline{\widetilde{S}}_j}\, \mathbf{1}_{\{(1+\delta)\overline{\widetilde{S}}_i-\widetilde{S}_i<a(n-i)^{1/3},\,\forall i<j,\,(1+\delta)\overline{\widetilde{S}}_j-\widetilde{S}_j\geq a(n-j)^{1/3}\}}\Big) \\
&\leq \mathrm{e}^{-a(n-j)^{1/3}}\, \mathbf{P}\Big((1+\delta)\overline{\widetilde{S}}_i - \widetilde{S}_i < a(n-i)^{1/3}, \forall i < j\Big).
\end{aligned}$$

It follows that

$$\mathbf{E}(\varrho_n^{(1+\delta)\theta}) \le \sum_{j=1}^{n} \mathrm{e}^{-a(n-j)^{1/3}}\, \mathbf{P}\Big((1+\delta)\overline{\widetilde{S}}_i - \widetilde{S}_i < a(n-i)^{1/3},\, \forall i < j\Big).$$

We choose $a := (\frac{3\pi^2\widetilde{\sigma}^2}{2(1+\delta)^2})^{1/3} = \frac{\theta\alpha_\theta^{1/3}}{(1+\delta)^{2/3}}$. Applying Corollary 4.2.2 (ii) to $(\widetilde{S}_i)$, we get $\mathbf{E}(\varrho_n^{(1+\delta)\theta}) \le \mathrm{e}^{-(a+o(1))n^{1/3}}$, for $n \to \infty$. By Chebyshev's inequality and the Borel–Cantelli lemma, $\mathbf{P}$-almost surely for $n \to \infty$, $\varrho_n^{(1+\delta)\theta} \le \mathrm{e}^{-(a+o(1))n^{1/3}}$. Since δ can be arbitrarily small, this implies (4.3.1), and completes the proof of Theorem 2.5.5. $\square$

4.4 Proof of Theorem 2.5.4: upper bound

We prove that if $\psi(1) = \psi'(1) = 0$, then[2]

$$\limsup_{n\to\infty} \frac{\max_{0\le k\le n} |X_k|}{(\log n)^3} \le \frac{8}{3\pi^2\sigma^2}\,, \qquad \mathbb{P}\text{-a.s.}, \tag{4.4.1}$$

where $\sigma^2 := \mathbf{E}\{\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)}\}$.

Let, for any $n \ge 1$,

$$\beta_n := P_\omega\{\tau_n < T_{\overleftarrow{\varnothing}}\}, \tag{4.4.2}$$

where $\tau_n := \inf\{i \ge 1 : |X_i| = n\}$ is as before the first time that the walk reaches the n-th generation, whereas $T_{\overleftarrow{\varnothing}} := \inf\{i \ge 0 : X_i = \overleftarrow{\varnothing}\}$ is the first time that the walk hits $\overleftarrow{\varnothing}$. There is a simple relation between β_n and $\varrho_n := P_\omega\{\tau_n < \tau_0\}$.

Lemma 4.4.1 *Assume that the walk (X_n) is recurrent. We have, for all $n \ge 1$,*

$$\varrho_n \le \beta_n \le \frac{\varrho_n}{\omega(\varnothing, \overleftarrow{\varnothing})}. \tag{4.4.3}$$

Proof of Lemma 4.4.1. The first inequality is trivial. Let us prove the second. Let $T_\varnothing^{(0)} := 0$ and $T_\varnothing^{(k)} := \inf\{i > T_\varnothing^{(k-1)} :\ X_i = \varnothing\}$ (for $k \ge 1$). In words, $T_\varnothing^{(k)}$ is the k-th return time to the root $\varnothing$. [Thus $T_\varnothing^{(1)} = \tau_0$.] Since the walk is recurrent, each $T_\varnothing^{(k)}$ is well-defined.

Recall that β_n represents the probability that starting from the root, the walk visits generation n before hitting $\overleftarrow{\varnothing}$. By considering the number of returns to $\varnothing$ (which can be 0) by the walk before visiting generation n, we have

$$\beta_n = P_\omega\{\tau_n < T_{\overleftarrow{\varnothing}}\} = \sum_{k=0}^{\infty} P_\omega\Big\{T_\varnothing^{(0)} < T_\varnothing^{(1)} < \cdots < T_\varnothing^{(k)} < \tau_n < T_\varnothing^{(k+1)},\ \tau_n < T_{\overleftarrow{\varnothing}}\Big\}.$$

[2] On the set of extinction, the upper bound is, in fact, trivially true.

Applying the strong Markov property successively at $T_\varnothing^{(k)}, \cdots, T_\varnothing^{(1)}$, we see that the probability on the right-hand side equals $[P_\omega\{T_\varnothing^{(1)} < (\tau_n \wedge T_{\overleftarrow{\varnothing}})\}]^k \, P_\omega\{\tau_n < T_\varnothing^{(1)}\}$ (notation: $u \wedge v := \min\{u, v\}$). Therefore

$$\beta_n = \frac{P_\omega\{\tau_n < T_\varnothing^{(1)}\}}{1 - P_\omega\{T_\varnothing^{(1)} < (\tau_n \wedge T_{\overleftarrow{\varnothing}})\}} = \frac{\varrho_n}{1 - P_\omega\{\tau_0 < (\tau_n \wedge T_{\overleftarrow{\varnothing}})\}}.$$

Since $1 - P_\omega\{\tau_0 < (\tau_n \wedge T_{\overleftarrow{\varnothing}})\} \geq 1 - P_\omega\{\tau_0 < T_{\overleftarrow{\varnothing}}\} = \omega(\varnothing, \overleftarrow{\varnothing})$, this yields the lemma. □

We now turn to the proof of (4.4.1). Assume $\psi(1) = \psi'(1) = 0$. We claim that it suffices to prove that

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbf{E}(\beta_n) \leq -\Big(\frac{3\pi^2\sigma^2}{8}\Big)^{1/3}. \tag{4.4.4}$$

Indeed, if (4.4.4) holds, then by Chebyshev's inequality and the Borel–Cantelli lemma, for any $\varepsilon > 0$ and $\mathbf{P}$-almost surely all sufficiently large n, $\beta_n \leq \exp[-(1-\varepsilon)(\frac{3\pi^2\sigma^2}{8})^{1/3} n^{1/3}]$, which by Lemma 4.4.1 yields $\varrho_n \leq \exp[-(1-\varepsilon)(\frac{3\pi^2\sigma^2}{8})^{1/3} n^{1/3}]$. In view of Fact 4.0.6, we obtain (4.4.1).

It remains to prove (4.4.4). Let $a := (\frac{3\pi^2\sigma^2}{8})^{1/3}$ and $n \geq 1$. By (4.3.2) and Lemma 4.4.1,

$$\mathbf{E}(\beta_n) \leq \sum_{j=1}^n \mathbf{E}\Big(\sum_{|y|=j} \mathbf{1}_{E_0(y)} \, \mathrm{e}^{-\overline{V}(y)}\Big),$$

where

$$E_0(y) := \Big\{\overline{V}(y) - V(y) \geq a(n-j)^{1/3}\Big\} \cap \bigcap_{i=1}^{j-1} \Big\{\overline{V}(y_i) - V(y_i) < a(n-i)^{1/3}\Big\}.$$

Applying (4.1.3), this leads to (with $\overline{S}_j := \max_{1\leq i\leq j} S_i$ as before):

$$\begin{aligned}
\mathbf{E}(\beta_n) \;&\leq\; \sum_{j=1}^n \mathbf{E}\Big\{\mathrm{e}^{S_j} \mathbf{1}_{\{\overline{S}_j - S_j \geq a(n-j)^{1/3},\, \overline{S}_i - S_i < a(n-i)^{1/3},\, \forall i<j\}} \mathrm{e}^{-\overline{S}_j}\Big\} \\
&\leq\; \sum_{j=1}^n \mathrm{e}^{-a(n-j)^{1/3}} \, \mathbf{P}\Big\{\overline{S}_i - S_i < a(n-i)^{1/3},\ \forall i < j\Big\},
\end{aligned}$$

which, according to Corollary 4.2.2 (i), is bounded by $\exp[-(1+o(1))(\frac{3\pi^2\sigma^2}{8})^{1/3} n^{1/3}]$ for $n \to \infty$. This yields (4.4.4). □

4.5 Proof of Theorem 2.5.4: lower bound

We start by recalling a spinal decomposition for the branching random walk $(V(x))$. This decomposition has been used in the literature by many authors in various forms, going back at least to Kahane and Peyrière [50]. The material in this paragraph is borrowed from Lyons, Pemantle and Peres [67] and Lyons [65]. The starting point is to work on space of trees and make a change of probabilities; we refer to the aforementioned references for more precision.

Assume $\psi(1) = 0$, i.e., $\mathbf{E}\{\sum_{|x|=1} \mathrm{e}^{-V(x)}\} = 1$. Let

$$W_n := \sum_{|x|=n} \mathrm{e}^{-V(x)}, \qquad n \ge 0.$$

Clearly, (W_n) is a martingale with respect to the filtration $(\mathscr{F}_n)$, where $\mathscr{F}_n$ is the sigma-algebra generated by the branching random walk in the first n generations.

By Kolmogorov's extension theorem, there exists a probability $\mathbf{Q}$ on $\mathscr{F}_\infty$ (the sigma-algebra generated by the branching random walk) such that for any n,

$$\mathbf{Q}_{|\mathscr{F}_n} = W_n \bullet \mathbf{P}_{|\mathscr{F}_n}. \tag{4.5.1}$$

The law of the branching random walk under the new probability $\mathbf{Q}$ is called the law of a *size-biased branching random walk*. It is clear that the size-biased branching random walk survives with probability one.

There is a one-to-one correspondence between a branching random walk and a marked tree. On the enlarged probability space formed by marked trees with distinguished rays, we may construct a probability $\mathbf{Q}$ satisfying (4.5.1), an infinite ray $\{w_0 = \varnothing, w_1, ..., w_n, ..\}$ such that for any $n \ge 1$, $\overleftarrow{w_n} = w_{n-1}$ (recalling that $\overleftarrow{x}$ is the parent of x) and

$$\mathbf{Q}\Big\{ w_n = x \,\Big|\, \mathscr{F}_n \Big\} = \frac{\mathrm{e}^{-V(x)}}{W_n}, \qquad \forall\, |x| = n. \tag{4.5.2}$$

For any individual $x \neq \varnothing$, let

$$\Delta V(x) := V(x) - V(\overleftarrow{x}).$$

We write, for $k \ge 1$,

$$\mathscr{I}_k := \Big\{ x :\, |x| = k,\ \overleftarrow{x} = w_{k-1},\ x \neq w_k \Big\}. \tag{4.5.3}$$

In words, $\mathscr{I}_k$ is the set of children of w_{k-1} except w_k, or equivalently, the set of the brothers of w_k, and is possibly empty. Finally, let us introduce the following sigma-field:

$$\mathscr{G}_n := \sigma\Big\{ (\Delta V(x),\ x \in \mathscr{I}_k),\ V(w_k),\ w_k,\ \mathscr{I}_k,\ 1 \le k \le n \Big\}. \tag{4.5.4}$$

The promised spinal decomposition is as follows (xu denoting concatenation of x and u). Although it slightly differs from the spinal decomposition presented in Lyons [65], we feel free to omit the proof.

Proposition 4.5.1 *Assume $\psi(1) = 0$, and fix $n \ge 1$. Under probability $\mathbf{Q}$,*

(i) *the random variables $(\Delta V(w_k),\ \Delta V(x),\ x \in \mathscr{I}_k)$, $1 \le k \le n$, are i.i.d.;*

(ii) *conditionally on $\mathscr{G}_n$, the shifted branching random walks $(\{V(xu) - V(x)\}_{|u|=k}, 0 \le k \le n - |x|)$, for $x \in \bigcup_{k=1}^n \mathscr{I}_k$, are independent, and have the same law as $(\{V(u)\}_{|u|=k}, 0 \le k \le n - |x|)$ under $\mathbf{P}$.*

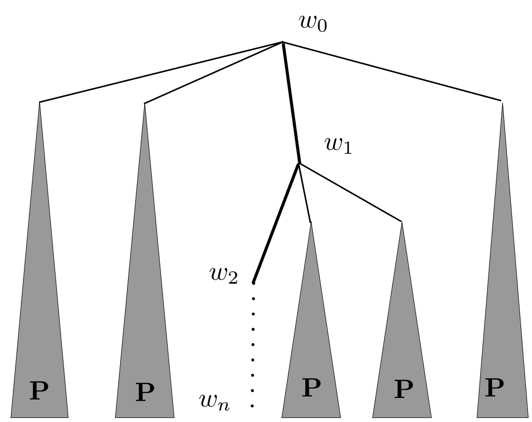


Figure 4.3: A $\mathbf{Q}$-tree

We now proceed to (the beginning of) the proof of the lower bound in Theorem 2.5.4, of which we recall the statement: under the assumption $\psi(1) = \psi'(1) = 0$, we have, on the set of non-extinction,

$$\liminf_{n\to\infty} \frac{\max_{0\le k\le n} |X_k|}{(\log n)^3} \ge \frac{4}{\alpha} = \frac{8}{3\pi^2\sigma^2}\,, \qquad \mathbb{P}\text{-a.s.}, \tag{4.5.5}$$

where $\sigma^2 := \mathbf{E}\{\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)}\}$.

Let $\beta_n := P_\omega\{\tau_n < T_{\overleftarrow{\varnothing}}\}$ be as in (4.4.2), where $\tau_n = \inf\{i \ge 1 : |X_i| = n\}$, and $T_{\overleftarrow{\varnothing}} = \inf\{i \ge 0 : X_i = \overleftarrow{\varnothing}\}$. We claim that it suffices to prove that

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbf{E}(\beta_n) \ge -\Big(\frac{3\pi^2\sigma^2}{8}\Big)^{1/3}. \tag{4.5.6}$$

It is indeed easy to check that (4.5.6) implies (4.5.5): Let $\mathscr{S} := \{\text{the system survives}\}$, $\mathscr{S}_n := \{\text{the system survives at least until generation } n\}$. Clearly $\mathscr{S} \subset \mathscr{S}_n$ for any n. Recall that there exists (see [45], p. 755) a constant $c > 0$ such that for all large n,

$$\mathbf{P}(W_n < n^{-c} \,|\, \mathscr{S}_n) \le n^{-2}.$$

On the other hand, we have (see [43], p. 543, Remark; the result therein states for the regular tree, but the same proof by convexity obviously holds in the general case)

$$\mathbf{E}\Big(\mathrm{e}^{-t\frac{\beta_n}{\mathbf{E}(\beta_n)}}\Big) \le \mathbf{E}(\mathrm{e}^{-tW_n}), \qquad t \ge 0.$$

Since $\beta_n = 0 = W_n$ on $\mathscr{S}_n^c$, it is equivalent to say that $\mathbf{E}(\mathrm{e}^{-t\frac{\beta_n}{\mathbf{E}(\beta_n)}} \,|\, \mathscr{S}_n) \le \mathbf{E}(\mathrm{e}^{-tW_n} \,|\, \mathscr{S}_n)$. Therefore, for any $\varepsilon > 0$ and all sufficiently large n,

$$\mathbf{P}\Big(\frac{\beta_n}{\mathbf{E}(\beta_n)} < \mathrm{e}^{-\varepsilon n^{1/3}} \,\Big|\, \mathscr{S}_n\Big) \le \mathrm{e}^1\, \mathbf{E}\Big(\mathrm{e}^{-\mathrm{e}^{\varepsilon n^{1/3}} W_n} \,\Big|\, \mathscr{S}_n\Big) \le n^{-2}\mathrm{e} + \mathrm{e}^{-n^{-c}\mathrm{e}^{\varepsilon n^{1/3}}}.$$

Since $\mathscr{S} \subset \mathscr{S}_n$, this implies $\sum_n \mathbf{P}(\frac{\beta_n}{\mathbf{E}(\beta_n)} < \mathrm{e}^{-\varepsilon n^{1/3}} \,|\, \mathscr{S}) < \infty$. If (4.5.6) holds, then by the Borel–Cantelli lemma, on the set $\mathscr{S}$, $\mathbf{P}$-almost surely for all sufficiently large n, $\beta_n \ge \mathrm{e}^{-\varepsilon n^{1/3}}\mathbf{E}(\beta_n) \ge \exp\{-[2\varepsilon + (\frac{3\pi^2\sigma^2}{8})^{1/3}]n^{1/3}\}$, and thus $\varrho_n \ge \omega(\varnothing, \overleftarrow{\varnothing}) \exp\{-[2\varepsilon + (\frac{3\pi^2\sigma^2}{8})^{1/3}]n^{1/3}\}$ (Lemma 4.4.1). In view of Fact 4.0.6, we obtain (4.5.5), the lower bound in Theorem 2.5.4.

The rest of the section is devoted to the proof of (4.5.6). Let as before $\varrho_n := P_\omega\{\tau_n < \tau_0\}$. Since $\beta_n \ge \varrho_n$ (Lemma 4.4.1), we only need to bound $\mathbf{E}(\varrho_n)$ from below.

For any vertex x, let P_ω^x be the (quenched) probability such that $P_\omega^x\{X_0 = x\} = 1$. We first prove a formula for ϱ_n without the assumption $\psi(1) = \psi'(1) = 0$. We mention that if $|x| = n$, then under P_ω^x, τ_n is the first *return* time to generation n.

Lemma 4.5.2 *Assume that the walk (X_n) is recurrent. For any $n \ge 1$, we have*

$$\varrho_n = \omega(\varnothing, \overleftarrow{\varnothing}) \sum_{|x|=n} \frac{\mathrm{e}^{-V(x)}}{\omega(x, \overleftarrow{x})} P_\omega^x\{\tau_n > \tau_0\}.$$

Proof of Lemma 4.5.2. The beginning of the proof uses a similar idea as in the proof of Lemma 4.4.1, except that instead of considering the number of returns to $\varnothing$ before hitting generation n, we consider the last site at generation n visited by the walk during an excursion. More precisely, for any x with $|x| \ge 1$, let $T_x^{0)} := 0$ and $T_x^{(k)} := \inf\{i > T_x^{(k-1)} :\ X_i = x\}$ (for $k \ge 1$). In words, $T_x^{(k)}$ is the time of the k-th visit at x.

Recall that ϱ_n is the (quenched) probability that during an excursion away from the root $\varnothing$, the walk hits generation n. By considering the last site at generation n visited by the walk during the excursion, we have

$$\begin{aligned}\varrho_n &= \sum_{|x|=n}\sum_{k=1}^{\infty} P_\omega\Big\{T_x^{(k)} < \tau_0 < T_x^{(k+1)},\ \max_{T_x^{(k)}<i\le\tau_0} |X_i| < n\Big\}\\ &= \sum_{|x|=n}\sum_{k=1}^{\infty} P_\omega\Big\{T_x^{(k)} < \tau_0,\ \max_{T_x^{(k)}<i\le\tau_0} |X_i| < n\Big\}.\end{aligned}$$

Applying the strong Markov property at $T_x^{(k)}$, we see that the probability on the right-hand side equals $P_\omega\{T_x^{(k)} < \tau_0\}\, P_\omega^x\{\tau_n > \tau_0\}$. Therefore,

$$\varrho_n = \sum_{|x|=n} P_\omega^x\{\tau_n > \tau_0\} \sum_{k=1}^{\infty} P_\omega\{T_x^{(k)} < \tau_0\} = \sum_{|x|=n} P_\omega^x\{\tau_n > \tau_0\} E_\omega\Big(\sum_{i=0}^{\tau_0-1} \mathbf{1}_{\{X_i=x\}}\Big).$$

Of course, $E_\omega(\sum_{i=0}^{\tau_0-1}\mathbf{1}_{\{X_i=x\}})$, being the expected number of visits at site x in an excursion, is a constant multiple of $\pi(x)$, if π is an invariant measure for the Markov chain (X_i). It is easily checked that if we take $\pi(x) := \frac{1}{\omega(x,\, \overleftarrow{x})}\,\mathrm{e}^{-V(x)}$ (for $x \ne \overleftarrow{\varnothing}$), then π is indeed an invariant measure. Therefore, there exists $0 < c(\omega) < \infty$ such that

$$E_\omega\Big(\sum_{i=0}^{\tau_0-1} \mathbf{1}_{\{X_i=x\}}\Big) = \frac{c(\omega)}{\omega(x,\ \overleftarrow{x})}\,\mathrm{e}^{-V(x)}.$$

To determine the value of $c(\omega)$, we take $x := \varnothing$, to see that $c(\omega) = \omega(\varnothing,\, \overleftarrow{\varnothing})$. This yields the lemma. $\square$

Assume $\psi(1) = 0$. We make use of the size-biased branching random walk, and work under the new probability $\mathbf{Q}$. Recall the definitions of $\mathbf{Q}$ and w_n from (4.5.1) and (4.5.2), respectively. By Lemma 4.5.2,

$$\mathbf{E}(\varrho_n) = \mathbf{E_Q}\Big\{\frac{\omega(\varnothing,\ \overleftarrow{\varnothing})}{\omega(w_n,\ w_{n-1})} P_\omega^{w_n}\{\tau_n > \tau_0\}\Big\}.$$

We observe that

$$P_\omega^{w_n}\{\tau_n > \tau_0\} = \prod_{j=1}^{n} P_\omega^{w_j}\{\tau_n > T(w_{j-1})\} =: \prod_{j=1}^{n} Y_j.$$

Obviously, $Y_n = \omega(w_n,\, w_{n-1})$, $Y_{n-1} = \omega(w_{n-1},\, w_{n-2})$.

Let $j \le n-2$. By the Markov property, $Y_j = \omega(w_j,\, w_{j-1}) + \sum_{x:\, \overleftarrow{x}=w_j} \omega(w_j,\, x) P_\omega^x\{\tau_n > T(w_{j-1})\}$, whereas by the strong Markov property, $P_\omega^x\{\tau_n > T(w_{j-1})\} = P_\omega^x\{\tau_n > T(w_j)\}\, Y_j$ for all x such that $\overleftarrow{x} = w_j$. Accordingly,

$$Y_j = \frac{\omega(w_j,\, w_{j-1})}{1 - \sum_{x:\, \overleftarrow{x}=w_j} \omega(w_j,\, x)\, P_\omega^x\{\tau_n > T(w_j)\}} = \frac{1}{1 + \sum_{x:\, \overleftarrow{x}=w_j} B(x) P_\omega^x\{\tau_n < T(w_j)\}},$$

where

$$B(x) := \mathrm{e}^{-[V(x)-V(\overleftarrow{x})]} = \frac{\omega(\overleftarrow{x},\, x)}{\omega(\overleftarrow{x},\, \overleftarrow{\overleftarrow{x}})}.$$

So, if we write

$$\xi_j := \sum_{x:\, \overleftarrow{x}=w_j,\, x\neq w_{j+1}} B(x) P_\omega^x\{\tau_n < T(w_j)\}, \qquad 1 \le j \le n-2,$$

then $Y_j = \frac{1}{1+\xi_j+(1-Y_{j+1})B(w_{j+1})}$, $1 \le j \le n-2$, and $\mathbf{E}(\varrho_n) = \mathbf{E}_{\mathbf{Q}}\{\frac{\omega(\varnothing,\, \overleftarrow{\varnothing})}{\omega(w_n,\, w_{n-1})} \prod_{j=1}^n Y_j\} = \mathbf{E}_{\mathbf{Q}}\{\omega(\varnothing,\, \overleftarrow{\varnothing}) \prod_{j=1}^{n-1} Y_j\}$.

Let $\mathscr{G}_n$ be the sigma-algebra generated by the first n generations of the spine (see (4.5.4)). By Proposition 4.5.1, under $\mathbf{Q}$, the random variables $\xi_1, \cdots, \xi_{n-1}$ are conditionally independent given $\mathscr{G}_n$. Moreover, for any $1 \le j \le n-2$,

$$\mathbf{E}_{\mathbf{Q}}(\xi_j \,|\, \mathscr{G}_n) = \sum_{x:\, \overleftarrow{x}=w_j,\, x\neq w_{j+1}} B(x)\, \mathbf{E}(\beta_{n-1-j}) \le \frac{\mathbf{E}(\beta_{n-1-j})}{\omega(w_j, w_{j-1})}. \tag{4.5.7}$$

We now provide a lower bound for $\mathbf{E}(\varrho_n)$, by replacing $(Y_j)_{1\le j\le n-1}$ by a new collection of random variables, denoted by $(Z_j)_{1\le j\le n-1}$ and defined as follows: $Z_{n-1} := Y_{n-1} = \omega(w_{n-1},\, w_{n-2})$ and for $1 \le j \le n-2$,

$$Z_j := \frac{1}{1 + \mathbf{E}_{\mathbf{Q}}(\xi_j \,|\, \mathscr{G}_n) + (1 - Z_{j+1})B(w_{j+1})}. \tag{4.5.8}$$

Since Z_{n-1}, $B(w_{n-1})$, $B(w_{n-2})$, $\cdots$, $B(w_1)$ are $\mathscr{G}_n$-measurable, it follows by induction on j that each Z_j, for $1 \le j \le n-1$, is $\mathscr{G}_n$-measurable.

Lemma 4.5.3 *Assume $\psi(1) = 0$. For any $n \ge 3$, we have*

$$\mathbf{E}_{\mathbf{Q}}\Big\{ \prod_{j=1}^{n-1} Y_j \,\Big|\, \mathscr{G}_n \Big\} \ge \prod_{j=1}^{n-1} Z_j, \qquad \mathbf{Q}\text{-a.s.}$$

Proof of Lemma 4.5.3. For any $c \in [0, 1]$ and $a := (a_1, \cdots, a_{n-1}) \in \mathbb{R}_+^{n-1}$, we define $F_{n-1}^{c,a}(u_{n-1}) := c$, $u_{n-1} \in \mathbb{R}_+$, and for $1 \le j \le n-2$,

$$F_j^{c,a}(u_j, \cdots, u_{n-2}) := \frac{1}{1 + u_j + a_{j+1}[1 - F_{j+1}^{c,a}(u_{j+1}, \cdots, u_{n-2})]}, \quad (u_j, \cdots, u_{n-2}) \in \mathbb{R}_+^{n-j-1}.$$

Then for $1 \le j \le n-1$, we have

$$Y_j = F_j^{Y_{n-1}, B(w)}(\xi_j, \cdots, \xi_{n-2}), \qquad Z_j = F_j^{Z_{n-1}, B(w)}(\mathbf{E_Q}(\xi_j \,|\, \mathscr{G}_n), \cdots, \mathbf{E_Q}(\xi_{n-2} \,|\, \mathscr{G}_n)),$$

where $B(w) := (B(w_1), \cdots, B(w_{n-1}))$. Note that both Y_{n-1} and $B(w)$ are $\mathscr{G}_n$-measurable.

Recall that (under $\mathbf{Q}$) ξ_1, $\cdots$, ξ_{n-2} are conditionally independent given $\mathscr{G}_n$. By Jensen's inequality, if $\Phi : \mathbb{R}_+^{n-2} \to \mathbb{R}$ is coordinate-wise convex, then $\mathbf{E_Q}\{\Phi(\xi_1, \cdots, \xi_{n-2}) \,|\, \mathscr{G}_n)\} \ge \Phi(\mathbf{E_Q}(\xi_1 \,|\, \mathscr{G}_n), \cdots, \mathbf{E_Q}(\xi_{n-2} \,|\, \mathscr{G}_n))$, $\mathbf{Q}$-a.s. So we only need to show that for any $c \in [0, 1]$ and $a \in \mathbb{R}_+^{n-1}$, $(u_1, \cdots, u_{n-2}) \mapsto \prod_{j=1}^{n-1} F_j^{c,a}(u_j, \cdots, u_{n-2})$ as a function on $\mathbb{R}_+^{n-2}$, is convex in each of u_i.

Since the product of non-negative, coordinate-wise non-increasing, coordinate-wise convex functions is still (non-negative, coordinate-wise non-increasing, and) coordinate-wise convex, we only have to check that for any $j \le n-2$, the function $(u_j, \cdots, u_{n-2}) \mapsto F_j^{c,a}(u_j, \cdots, u_{n-2})$ is non-negative (which is obvious), coordinate-wise non-increasing, and coordinate-wise convex. We prove it by induction on j.

By definition, $F_{n-2}^{c,a}(u_{n-2}) = [1 + u_{n-2} + (1-c)a_{n-1}]^{-1}$, which is obviously non-increasing and convex in u_{n-2}.

Assume that for $1 \le j \le n-3$, $(u_{j+1}, \cdots, u_{n-2}) \mapsto F_{j+1}^{c,a}(u_{j+1}, \cdots, u_{n-2})$ is coordinate-wise non-increasing and coordinate-wise convex. Since

$$F_j^{c,a}(u_j, \cdots, u_{n-2}) = \frac{1}{1 + u_j + a_{j+1}[1 - F_{j+1}^{c,a}(u_{j+1}, \cdots, u_{n-2})]},$$

$F_j^{c,a}$ is non-increasing and convex in each of u_i (for $j \le i \le n-2$): the monotonicity is obvious, whereas the convexity follows from the fact that $y \mapsto \frac{1}{1+u_j+(1-y)a_{j+1}}$ is convex and non-decreasing on $[0, 1]$ and that $f \circ g$ is convex if f is convex and non-decreasing while g is convex. □

Recall that $\mathbf{E}(\varrho_n) = \mathbf{E_Q}\{\omega(\varnothing, \overleftarrow{\varnothing}) \prod_{j=1}^{n-1} Y_j\}$. Since $\omega(\varnothing, \overleftarrow{\varnothing})$ is $\mathscr{G}_n$-measurable, it follows from Lemma 4.5.3 that

$$\mathbf{E}(\varrho_n) \ge \mathbf{E_Q}\Big\{\omega(\varnothing, \overleftarrow{\varnothing}) \prod_{j=1}^{n-1} Z_j\Big\}. \tag{4.5.9}$$

We now give a lower bound for $\prod_{j=1}^{n-1} Z_j$ by means of a deterministic lemma. The proof of the lemma is in section 4.6

Lemma 4.5.4 *Let $n > k \geq 2$. Let $b_{j+1} > 0$ and $r_j \geq 0$ for all $0 \leq j < n$. Define $(z_j)_{0\leq j\leq n}$ by $z_n = 0$ and*

$$z_j := \frac{1}{1 + r_j + b_{j+1}(1 - z_{j+1})}, \qquad 0 \leq j \leq n-1.$$

Let $v(0) := 0$ and $v(j) := -\sum_{i=1}^{j} \log b_i$, $1 \leq j \leq n$. For any $m_0 = 0 < m_1 < ... < m_k = n-1$, we have

$$\prod_{j=1}^{n-1} z_j \geq \frac{2^{-k}}{\prod_{i=1}^{k}(m_i - m_{i-1})} \exp\Big\{ - \sum_{i=1}^{k} \Big(\lambda_i + (m_i - m_{i-1})^2 \, r^{(i)} \, \mathrm{e}^{v_i^*}\Big)\Big\},$$

where for $1 \leq i \leq k$ (with $y^+ := \max\{y, 0\}$ for $y \in \mathbb{R}$),

$$\begin{aligned} r^{(i)} &:= \max_{m_{i-1}<j\leq m_i} r_j, \\ \lambda_i &:= \max_{m_{i-1}<j\leq m_i} (v(j) - v(m_i)) + (v(m_i) - v(1+m_i))^+, \\ v_i^* &:= \max_{m_{i-1}<j\leq \ell\leq m_i} (v(j) - v(\ell)). \end{aligned}$$

We continue with the proof of the lower bound in Theorem 2.5.4. Recall from (4.5.9) that $\mathbf{E}(\varrho_n) \geq \mathbf{E_Q}\{\omega(\varnothing, \overleftarrow{\varnothing}) \prod_{j=1}^{n-1} Z_j\}$.

Let $k \geq 2$ and $m_0 := 0 < m_1 < m_2 < ... < m_k = n-1$. By applying Lemma 4.5.4 to $b_{j+1} = B(w_{j+1})$ and $r_j := \mathbf{E_Q}(\xi_j \,|\, \mathscr{G}_n)$, and arguing that $\prod_{i=1}^{k}(m_i - m_{i-1}) \leq \prod_{i=1}^{k} n = n^k$, we obtain:

$$\mathbf{E}(\varrho_n) \geq \frac{1}{(2n)^k} \mathbf{E_Q}\Big(\omega(\varnothing, \overleftarrow{\varnothing}) \, \mathrm{e}^{-\sum_{i=1}^{k} \Lambda_i - \sum_{i=1}^{k} (m_i - m_{i-1})^2 \, r^{(i)} \, \mathrm{e}^{S_i^*}}\Big), \tag{4.5.10}$$

where, for any $1 \leq i \leq k$,

$$\begin{aligned} r^{(i)} &:= \max_{m_{i-1}<j\leq m_i} \mathbf{E_Q}(\xi_j \,|\, \mathscr{G}_n) \leq \max_{m_{i-1}<j\leq m_i} \frac{\mathbf{E}(\beta_{n-1-j})}{\omega(w_j, w_{j-1})}, \\ \Lambda_i &:= \max_{m_{i-1}<j\leq m_i} (S_j - S_{m_i}) + (S_{m_i} - S_{1+m_i})^+, \\ S_i^* &:= \max_{m_{i-1}<j\leq \ell\leq m_i} (S_j - S_\ell), \end{aligned}$$

with $S_j := V(w_j)$, $0 \leq j \leq n$. [In the inequality for $r^{(i)}$, we used (4.5.7).]

We choose: $\chi := \frac{1}{100}$, $k := \lfloor n^{\frac{1-\chi}{3}} \rfloor$, $m_0 := 0$, $m_i := n - (k-i)^3 \lfloor n^\chi \rfloor$ for $1 \leq i \leq k-1$, and $m_k := n-1$.

Let $c > 1$ be a constant sufficiently large such that $\mathbf{Q}\{S_2 \ge S_1,\ \omega(w_1,\, \varnothing) \ge \frac{1}{c}\} > \frac{1}{c}$. Let

$$E_n^{(1)} := \left\{ S_{j+1} \ge S_j,\, \omega(w_j, w_{j-1}) \ge \frac{1}{c},\, \forall m_{k-1}+1 \le j \le m_k \right\}.$$

On $E_n^{(1)}$, we have $\Lambda_k \le 0$, $r^{(k)} \le c$, and $S_k^* = 0$, whereas by definition, $m_k - m_{k-1} = \lfloor n^\chi \rfloor - 1 \le n^\chi$. Therefore, by (4.5.10),

$$\begin{aligned} \mathbf{E}(\varrho_n) &\ge \frac{\mathrm{e}^{-c\, n^{2\chi}}}{(2n)^k}\, \mathbf{E}_{\mathbf{Q}}\Big(\omega(\varnothing,\, \overleftarrow{\varnothing})\, \mathrm{e}^{-\sum_{i=1}^{k-1} \Lambda_i - \sum_{i=1}^{k-1} (m_i - m_{i-1})^2\, r^{(i)}\, \mathrm{e}^{S_i^*}}\, \mathbf{1}_{E_n^{(1)}} \Big) \\ &= \frac{\mathrm{e}^{-c\, n^{2\chi}}}{(2n)^k}\, \mathbf{E}_{\mathbf{Q}}\Big(\omega(\varnothing,\, \overleftarrow{\varnothing})\, \mathrm{e}^{-\sum_{i=1}^{k-1} \Lambda_i - \sum_{i=1}^{k-1} (m_i - m_{i-1})^2\, r^{(i)}\, \mathrm{e}^{S_i^*}} \Big)\, \mathbf{Q}(E_n^{(1)}), \end{aligned}$$

the last identity being a consequence of the fact (notation: $w_{-1} := \overleftarrow{\varnothing}$) that under $\mathbf{Q}$, $(S_j - S_{j-1},\, \omega(w_{j-1}, w_{j-2}))$, for $j \ge 1$, are independent (they are i.i.d. for $j \ge 2$). By the definition of c, $\mathbf{Q}(E_n^{(1)}) = [\mathbf{Q}\{S_2 \ge S_1,\ \omega(w_1,\, \varnothing) \ge \frac{1}{c}\}]^{m_k - m_{k-1}} \ge (\frac{1}{c})^{m_k - m_{k-1}} = (\frac{1}{c})^{\lfloor n^\chi \rfloor - 1}$. Hence,

$$\mathbf{E}(\varrho_n) \ge \frac{\mathrm{e}^{-c\, n^{2\chi}}}{(2n)^k\, c^{\lfloor n^\chi \rfloor - 1}}\, \mathbf{E}_{\mathbf{Q}}\Big(\omega(\varnothing,\, \overleftarrow{\varnothing})\, \mathrm{e}^{-\sum_{i=1}^{k-1} \Lambda_i - \sum_{i=1}^{k-1} (m_i - m_{i-1})^2\, r^{(i)}\, \mathrm{e}^{S_i^*}} \Big).$$

Let $\varepsilon \in (0,\, \frac{\chi}{3})$. Write $a_* := (\frac{3\pi^2\sigma^2}{8})^{1/3}$. By (4.4.4), there exists some constant $c_1 > 0$ such that $\mathbf{E}(\beta_i) \le c_1\, \mathrm{e}^{-(a_* - \varepsilon)(i+1)^{1/3}}$ for all $i \ge 1$. Thus

$$r^{(i)} \le \frac{c_1\, \mathrm{e}^{-(a_* - \varepsilon)(n - m_i)^{1/3}}}{\min_{m_{i-1} < j \le m_i} \omega(w_j,\, w_{j-1})}, \qquad 1 \le i \le k.$$

Consider

$$E_n^{(2)} := \big\{ \omega(w_j, w_{j-1}) \ge \mathrm{e}^{-n^\varepsilon},\, \forall\, 1 \le j \le m_{k-1} \big\} \cap \{ \omega(\varnothing,\, \overleftarrow{\varnothing}) \ge \mathrm{e}^{-n^\varepsilon} \}.$$

On $E_n^{(2)}$, we have, for any $1 \le i \le k-1$, $r^{(i)} \le c_1 \mathrm{e}^{-(a_* - \varepsilon)(n - m_i)^{1/3} + n^\varepsilon}$, whereas $m_i - m_{i-1} \le n$, thus $(m_i - m_{i-1})^2 r^{(i)} \le \mathrm{e}^{-(a_* - 2\varepsilon)(n - m_i)^{1/3}}$ (for all sufficiently large n). Hence

$$\mathbf{E}(\varrho_n) \ge \frac{\mathrm{e}^{-c\, n^{2\chi} - n^\varepsilon}}{(2n)^k\, c^{\lfloor n^\chi \rfloor - 1}}\, \mathbf{E}_{\mathbf{Q}}\Big(\mathrm{e}^{-\sum_{i=1}^{k-1} [\Lambda_i + \mathrm{e}^{S_i^* - (a_* - 2\varepsilon)(n - m_i)^{1/3}}]}\, \mathbf{1}_{E_n^{(2)}} \Big). \tag{4.5.11}$$

Let, for $1 \le i \le k-1$,

$$E_{n,i}^{(3)} := \Big\{ S_i^* < (a_* - 2\varepsilon)(n - m_i)^{1/3},\ \max_{m_{i-1} < j \le m_i} (S_j - S_{m_i}) \le n^\varepsilon,\ |S_{1+m_i} - S_{m_i}| \le n^\varepsilon \Big\}.$$

On the event $E_{n,i}^{(3)}$ (for $1 \le i \le k-1$), we have $\Lambda_i \le n^\varepsilon + n^\varepsilon = 2n^\varepsilon$, and, of course, $S_i^* - (a_* - 2\varepsilon)(n-m_i)^{1/3} \le 0$, so that $\Lambda_i + \mathrm{e}^{S_i^* - (a_*-2\varepsilon)(n-m_i)^{1/3}} \le 2n^\varepsilon + 1 \le 3n^\varepsilon$. Going back to (4.5.11), we obtain:

$$\mathbf{E}(\varrho_n) \ge \frac{\mathrm{e}^{-c\,n^{2\chi} - n^\varepsilon - 3n^\varepsilon(k-1)}}{(2n)^k\, c^{\lfloor n^\chi \rfloor - 1}}\, \mathbf{Q}\Big(E_n^{(2)} \cap \bigcap_{i=1}^{k-1} E_{n,i}^{(3)} \Big). \tag{4.5.12}$$

By independence,

$$\begin{aligned} \mathbf{Q}\Big(E_n^{(2)} \cap \bigcap_{i=1}^{k-1} E_{n,i}^{(3)} \Big) &= \mathbf{Q}\{\omega(\varnothing, \overleftarrow{\varnothing}) \ge \mathrm{e}^{-n^\varepsilon}\} \prod_{i=1}^{k-1} \mathbf{Q}\Big(E_{n,i}^{(3)}, \min_{m_{i-1} < \ell \le m_i} \omega(w_\ell, w_{\ell-1}) \ge \mathrm{e}^{-n^\varepsilon} \Big) \\ &\ge \frac{1}{2} \prod_{i=1}^{k-1} \mathbf{Q}\Big(E_{n,i}^{(3)}, \min_{m_{i-1} < \ell \le m_i} \omega(w_\ell, w_{\ell-1}) \ge \mathrm{e}^{-n^\varepsilon} \Big), \end{aligned}$$

the last inequality holding for all sufficiently large n (in view of the fact that $\mathbf{Q}\{\omega(\varnothing, \overleftarrow{\varnothing}) \ge \mathrm{e}^{-n^\varepsilon}\} \to 1$, $n \to \infty$). By independence again, for any $1 \le i \le k-1$,

$$\begin{aligned} & \mathbf{Q}\Big(E_{n,i}^{(3)}, \min_{m_{i-1} < \ell \le m_i} \omega(w_\ell, w_{\ell-1}) \ge \mathrm{e}^{-n^\varepsilon} \Big) \\ = \;& \mathbf{Q}\Big(|S_{1+m_i} - S_{m_i}| \le n^\varepsilon,\ \omega(w_{m_i}, w_{m_i-1}) \ge \mathrm{e}^{-n^\varepsilon} \Big) \times \\ & \times \mathbf{Q}\Big(S_i^* < (a_* - 2\varepsilon)(n-m_i)^{1/3},\ \max_{m_{i-1} < j \le m_i} (S_j - S_{m_i}) \le n^\varepsilon, \\ & \qquad\qquad \min_{m_{i-1} < \ell < m_i} \omega(w_\ell, w_{\ell-1}) \ge \mathrm{e}^{-n^\varepsilon} \Big) \\ = \;& \mathbf{Q}\Big(|S_2 - S_1| \le n^\varepsilon,\ \omega(w_1, \varnothing) \ge \mathrm{e}^{-n^\varepsilon} \Big) \times \mathbf{Q}\Big(F_i(n), \min_{1 \le \ell < \Delta_i} \omega(w_\ell, w_{\ell-1}) \ge \mathrm{e}^{-n^\varepsilon} \Big), \end{aligned}$$

where, for $1 \le i \le k-1$,

$$\begin{aligned} \Delta_i &:= m_i - m_{i-1}, \\ F_i(n) &:= \Big\{ \max_{1 \le \ell \le \Delta_i} (\overline{S}_\ell - S_\ell) < (a_* - 2\varepsilon)(n-m_i)^{1/3},\ \overline{S}_{\Delta_i} - S_{\Delta_i} \le n^\varepsilon \Big\}, \end{aligned}$$

with $\overline{S}_\ell := \max_{1 \le j \le \ell} S_j$ as before. Again, $\mathbf{Q}\{|S_2 - S_1| \le n^\varepsilon,\ \omega(w_1, \varnothing) \ge \mathrm{e}^{-n^\varepsilon}\}$ is greater than $\frac{1}{2}$ for large n because it converges to 1. Therefore, for all large n,

$$\mathbf{Q}\Big(E_n^{(2)} \cap \bigcap_{i=1}^{k-1} E_{n,i}^{(3)} \Big) \ge \frac{1}{4} \prod_{i=1}^{k-1} \mathbf{Q}\Big(F_i(n), \min_{1 \le \ell < \Delta_i} \omega(w_\ell, w_{\ell-1}) \ge \mathrm{e}^{-n^\varepsilon} \Big).$$

To bound the probability expression on the right-hand side, we use the following lemma which is a uniform version of Proposition 4.2.1. Its proof is in Section 4.6.

Lemma 4.5.5 *Let $S_i - S_{i-1}$, $i \ge 1$, be i.i.d. mean-zero random variables ($S_0 := 0$) with $\sigma^2 := \mathbf{E}(S_1^2) \in (0, \infty)$. For any $\delta > 0$, there exist $m_0 > 1$ and $0 < \eta < 1$ such that for all $m_0 < m < \eta \sqrt{n}$, for all events $A_i^{(n)}$, $1 \le i \le n$, satisfying the following two conditions:*

- *$(S_i - S_{i-1},\, A_i^{(n)})$, for $1 \le i \le n$, are i.i.d.,*
- *$\mathbf{P}(\cap_{i=1}^n A_i^{(n)}) \ge 1 - \frac{\eta}{m}$,*

we have

$$\frac{\eta}{m}\, \mathrm{e}^{-(1+\delta)\frac{\pi^2\sigma^2}{8}\frac{n}{m^2}} \le \mathbf{P}\Big(\max_{1\le i\le n} (\overline{S}_i - S_i) < m,\, \overline{S}_n = S_n,\, \bigcap_{i=1}^n A_i^{(n)} \Big) \le \mathrm{e}^{-(1-\delta)\frac{\pi^2\sigma^2}{8}\frac{n}{m^2}},$$

where $\overline{S}_i := \max_{1\le j\le i} S_j$.

Since $\mathbf{E}[\frac{1}{\omega(\varnothing,\, \overleftarrow{\varnothing})}] = \mathbf{E}[1 + \sum_{i=1}^N A_i] = 1 + \mathrm{e}^{\psi(1)} = 2$, we have, $\mathbf{Q}\{\omega(w_1, \varnothing) < \mathrm{e}^{-n^\varepsilon}\} = \mathbf{P}\{\omega(\varnothing, \overleftarrow{\varnothing}) < \mathrm{e}^{-n^\varepsilon}\} \le 2\mathrm{e}^{-n^\varepsilon}$ (by Markov's inequality). Therefore, for all large n and all $1 \le i \le k-1$,

$$\begin{aligned} \mathbf{Q}\Big\{ \min_{1\le \ell<\Delta_i} \omega(w_\ell, w_{\ell-1}) \ge \mathrm{e}^{-n^\varepsilon} \Big\} &= [\mathbf{Q}\{\omega(w_1, \varnothing) \ge \mathrm{e}^{-n^\varepsilon}\}]^{\Delta_i - 1} \\ &\ge (1 - 2\mathrm{e}^{-n^\varepsilon})^{\Delta_i - 1} \\ &\ge (1 - 2\mathrm{e}^{-n^\varepsilon})^n \ge 1 - \mathrm{e}^{-n^{\varepsilon/2}}. \end{aligned}$$

We apply Lemma 4.5.5 to $A_i^{(n)} := \{\omega(w_i, w_{i-1}) \ge \mathrm{e}^{-n^\varepsilon}\}$ (with $\Delta_i - 1$ and $(n-m_i)^{1/3}$ playing the roles of n and m, respectively; noting that $\Delta_i - 1 \sim 3(k-i)^2 n^\chi$ and $(n-m_i)^{1/3} \sim (k-i)n^{\chi/3}$, so the last condition in the lemma on $A_i^{(n)}$ is satisfied), to see that for all large n and for all $1 \le i \le k-1$,

$$\mathbf{Q}\Big(F_i(n),\, \min_{1\le \ell<\Delta_i} \omega(w_\ell, w_{\ell-1}) \ge \mathrm{e}^{-n^\varepsilon} \Big) \ge \exp\Big(-(1+\varepsilon)\frac{3\pi^2\sigma^2}{8(a_* - 2\varepsilon)^2} n^{\chi/3} \Big),$$

which implies that

$$\mathbf{Q}\Big(E_n^{(2)} \cap \bigcap_{i=1}^{k-1} E_{n,i}^{(3)} \Big) \ge \frac{1}{4} \exp\Big(-(k-1)(1+\varepsilon)\frac{3\pi^2\sigma^2}{8(a_* - 2\varepsilon)^2} n^{\chi/3} \Big).$$

By definition, $k = \lfloor n^{(1-\chi)/3} \rfloor$; hence

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbf{Q}\Big(E_n^{(2)} \cap \bigcap_{i=1}^{k-1} E_{n,i}^{(3)} \Big) \ge -(1+\varepsilon)\frac{3\pi^2\sigma^2}{8(a_* - 2\varepsilon)^2}.$$

This, together with (4.5.12), yields

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbf{E}(\varrho_n) \ge -\Big(\frac{3\pi^2\sigma^2}{8} \Big)^{1/3}.$$

Since $\beta_n \ge \varrho_n$ (Lemma 4.4.1), we obtain (4.5.6), thus the lower bound in Theorem 2.5.4.

4.6 Proofs of Lemmas 4.5.4 and 4.5.5

Proof of Lemma 4.5.4. Although the lemma is deterministic, our proof is probabilistic. Let $(\eta_i)_{i\geq 0}$ be a Markov chain on $\{0, 1, \cdots, n\}$ with transition probabilities

$$\mathbf{P}\Big(\eta_{i+1} = k \,\Big|\, \eta_i = j\Big) = \begin{cases} \frac{b_{j+1}}{1+b_{j+1}}, & \text{if } k = j+1, \\ \\ \frac{1}{1+b_{j+1}}, & \text{if } k = j-1. \end{cases} \qquad 0 < j < n.$$

Define $\tau_\eta(j) := \inf\{i \geq 1 : \eta_i = j\}$. Let $\mathbf{P}_j$ be the probability such that $\mathbf{P}_j\{\eta_0 = j\} = 0$, and let $\mathbf{E}_j$ be the expectation with respect to $\mathbf{P}_j$. We claim that

$$\prod_{j=1}^{n-1} z_j \geq \prod_{i=1}^{k} \mathbf{E}_{m_i}\Big(\mathbf{1}_{(\tau_\eta(m_{i-1})<\tau_\eta(m_i))}(1 + r^{(i)})^{-\tau_\eta(m_{i-1})}\Big), \tag{4.6.1}$$

and for any integers $0 \leq \ell < m \leq n$ and $r \geq 0$,

$$\mathbf{E}_m\Big((1+r)^{-\tau_\eta(\ell)}\mathbf{1}_{(\tau_\eta(\ell)<\tau_\eta(m))}\Big) \;\geq\; \frac{1}{2(m-\ell)} \exp\Big\{ -\max_{\ell<i\leq m}(v(i) - v(m)) -(v(m) - v(m+1))^+ - r(m-\ell)^2 \mathrm{e}^{\max_{\ell<i\leq j\leq m}[v(i)-v(j)]}\Big\}. \tag{4.6.2}$$

Plainly Lemma 4.5.4 will follow from (4.6.1) and (4.6.2).

To prove (4.6.1), we consider a Markov chain $(\widetilde{\eta}_i)_{i\geq 0}$ on $\{0, 1, \cdots, n\}$, having an absorbing point ∂, such that

$$\mathbf{P}\Big(\widetilde{\eta}_{i+1} = k \,\Big|\, \widetilde{\eta}_i = j\Big) = \begin{cases} b_{j+1}\, q_j, & \text{if } k = j+1, \\ q_j, & \text{if } k = j-1, \\ r_j q_j, & \text{if } k = \partial, \end{cases} \qquad 0 < j < n,$$

with $q_j := \frac{1}{b_{j+1}+1+r_j}$ for all $0 < j < n$. Let $\tau_{\widetilde{\eta}}(j) := \inf\{i \geq 1 : \widetilde{\eta}_i = j\}$. Then

$$z_j = \mathbf{P}_j\Big(\tau_{\widetilde{\eta}}(j-1) < \tau_{\widetilde{\eta}}(n)\Big), \qquad \forall 1 \leq j \leq n-1.$$

Indeed, $z_{n-1} = q_{n-1}$, and if $z_j = \mathbf{P}_j(\tau_{\widetilde{\eta}}(j-1) < \tau_{\widetilde{\eta}}(n))$ for $j \in [2, n-1] \cap \mathbb{Z}$, then $\mathbf{P}_{j-1}(\tau_{\widetilde{\eta}}(j-2) < \tau_{\widetilde{\eta}}(n)) = q_{j-1} + b_j q_{j-1} \mathbf{P}_j(\tau_{\widetilde{\eta}}(j-2) < \tau_{\widetilde{\eta}}(n)) = q_{j-1} + b_j q_{j-1} z_j \mathbf{P}_{j-1}(\tau_{\widetilde{\eta}}(j-2) < \tau_{\widetilde{\eta}}(n))$ by the Markov property. Hence $\mathbf{P}_{j-1}(\tau_{\widetilde{\eta}}(j-2) < \tau_{\widetilde{\eta}}(n)) = \frac{q_{j-1}}{1-b_{j-1}q_{j-1}z_j} = \frac{1}{1+r_{j-1}+b_j(1-z_j)}$ proving that $\mathbf{P}_{j-1}(\tau_{\widetilde{\eta}}(j-2) < \tau_{\widetilde{\eta}}(n)) = z_{j-1}$. As a consequence,

$$\mathbf{P}_{n-1}\Big(\tau_{\widetilde{\eta}}(0) < \tau_{\widetilde{\eta}}(n)\Big) = \prod_{j=1}^{n-1} z_j. \tag{4.6.3}$$

We claim that for $0 \le \ell < m < n$,

$$\mathbf{P}_m\Big(\tau_{\widetilde{\eta}}(\ell) < \tau_{\widetilde{\eta}}(m)\Big) \ge \mathbf{E}_m\Big(\mathbf{1}_{(\tau_\eta(\ell)<\tau_\eta(m))}(1+r)^{-\tau_\eta(\ell)}\Big), \tag{4.6.4}$$

where $r := \max_{\ell<j\le m} r_j$. Since $\mathbf{P}_{n-1}(\tau_{\widetilde{\eta}}(0) < \tau_{\widetilde{\eta}}(n)) \ge \prod_{i=1}^k \mathbf{P}_{m_i}(\tau_{\widetilde{\eta}}(m_{i-1}) < \tau_{\widetilde{\eta}}(m_i))$, (4.6.1) will be a consequence of (4.6.3) and (4.6.4).

To prove (4.6.4), let $\Omega_{\ell,m}$ be the set of all (finite) paths of $\widetilde{\eta}$ starting from m and hitting ℓ before returning to m (and without being absorbed by ∂). For any $\gamma \in \Omega_{\ell,m}$, let $L_\gamma^\pm(j) := \sum_{i\ge 0} \mathbf{1}_{(\gamma_i=j,\,\gamma_{i+1}=j\pm 1)}$ and $L_\gamma(j) := L_\gamma^+(j) + L_\gamma^-(j)$. Then

$$\begin{aligned}
\mathbf{P}_m\Big(\tau_{\widetilde{\eta}}(\ell) < \tau_{\widetilde{\eta}}(m)\Big) &= \sum_{\gamma\in\Omega_{\ell,m}} \prod_{\ell<j\le m} (b_{j+1}q_j)^{L_\gamma^+(j)}(q_j)^{L_\gamma^-(j)} \\
&= \sum_{\gamma\in\Omega_{\ell,m}} \prod_{\ell<j\le m} (\frac{b_{j+1}}{1+b_{j+1}})^{L_\gamma^+(j)}(\frac{1}{1+b_{j+1}})^{L_\gamma^-(j)}\Big(\frac{1+b_{j+1}}{1+b_{j+1}+r_j}\Big)^{L_\gamma(j)} \\
&\ge \sum_{\gamma\in\Omega_{\ell,m}} \prod_{\ell<j\le m} (\frac{b_{j+1}}{1+b_{j+1}})^{L_\gamma^+(j)}(\frac{1}{1+b_{j+1}})^{L_\gamma^-(j)}\Big(\frac{1}{1+r}\Big)^{L_\gamma(j)} \\
&= \mathbf{E}_m\Big(\mathbf{1}_{(\tau_\eta(\ell)<\tau_\eta(m))}(1+r)^{-\tau_\eta(\ell)}\Big),
\end{aligned}$$

yielding (4.6.4) and hence (4.6.1).

It remains to show (4.6.2). Recall that $v(j) = -\sum_{i=1}^j \log b_i$ for $j \ge 1$. Then

$$\begin{aligned}
\mathbf{P}_m\Big(\tau_\eta(\ell) < \tau_\eta(m)\Big) &= \frac{1}{1+b_{m+1}}\mathbf{P}_{m-1}\Big(\tau_\eta(\ell) < \tau_\eta(m)\Big) \\
&= \frac{1}{1+b_{m+1}} \frac{\mathrm{e}^{v(m)}}{\sum_{i=\ell+1}^m \mathrm{e}^{v(i)}} \\
&\ge \frac{1}{2(m-\ell)} \exp\Big\{ - \max_{\ell<i\le m}(v(i)-v(m)) - (v(m)-v(m+1))^+\Big\}. \qquad (4.6.5)
\end{aligned}$$

Under $\mathbf{P}_m$ and conditionally on $\{\tau_\eta(\ell) < \tau_\eta(m)\}$, $\tau_\eta(\ell)$ is stochastically smaller than the hitting time of ℓ by a Markov chain with the same probability transition as η but reflecting on m. The expectation of the later hitting time was estimated by Golosov ([39], p. 498, (A.1)). Hence

$$\mathbf{E}_m\Big(\tau_\eta(\ell)\,\Big|\,\tau_\eta(\ell) < \tau_\eta(m)\Big) \le (m-\ell)^2 \exp\Big(\max_{\ell<i\le j\le m}(v(i)-v(j))\Big),$$

which, by means of the elementary inequality $(1+r)^{-u} \ge \mathrm{e}^{-ru}$ for $u \ge 0$ and Jensen's inequality, implies that

$$\mathbf{E}_m\Big((1+r)^{-\tau_\eta(\ell)}\Big) \ge \exp\Big(- r(m-\ell)^2 \mathrm{e}^{\max_{\ell<i\le j\le m}(v(i)-v(j))}\Big).$$

This together with (4.6.5) implies (4.6.2), completing the proof of Lemma 4.5.4. □

Proof of Lemma 4.5.5. We start with the proof of the lower bound. By monotonicity in m, we may assume that m is an integer. Let $a \geq 2$ be an integer whose value will be chosen later on. Let $\eta < \frac{1}{a}$. Let $K := \lfloor \frac{n}{am^2} \rfloor$, and $n_i = iam^2$ for $0 \leq i \leq K-1$ and $n_K := n$. Write $S_n^\# := \max_{1 \leq i \leq n}(\overline{S}_i - S_i)$. It is clear that

$$\begin{aligned}
&\mathbf{P}\Big(S_n^\# < m, \, \overline{S}_n = S_n, \bigcap_{i=1}^n A_i^{(n)}\Big) \\
\geq \ & \mathbf{P}\Big(\forall 1 \leq j \leq K, \max_{n_{j-1} < i \leq n_j} (\overline{S}_i - S_i) < m, \, \overline{S}_{n_j} - S_{n_j} < \delta m, \, \overline{S}_n = S_n, \bigcap_{i=1}^n A_i^{(n)}\Big).
\end{aligned}$$

For any $1 \leq j \leq K$, conditionally on $\sigma\{S_i, A_i^{(n)}, 1 \leq i \leq n_{j-1}\}$ and on $\{\overline{S}_{n_{j-1}} - S_{n_{j-1}} = x\}$, the reflecting random walk $(\overline{S}_{i+n_{j-1}} - S_{i+n_{j-1}}, \, 0 \leq i \leq n_j - n_{j-1})$ has the same law as $(\max\{x, \, \overline{S}_i\} - S_i, \, 0 \leq i \leq n_j - n_{j-1})$. Accordingly,

$$\mathbf{P}\Big(S_n^\# < m, \, \overline{S}_n = S_n, \bigcap_{i=1}^n A_i^{(n)}\Big) \geq q_{n,m}^{K-1} b_{n,m},$$

where

$$\begin{aligned}
q_{n,m} \ &:= \ \mathbf{P}\Big(S_{am^2}^\# < (1-\delta)m, \, \overline{S}_{am^2} - S_{am^2} < \delta m, \, \overline{S}_{am^2} > \delta m, \bigcap_{i=1}^{am^2} A_i^{(n)}\Big), \\
b_{n,m} \ &:= \ \mathbf{P}\Big(S_{n_K - n_{K-1}}^\# < (1-\delta)m, \, \overline{S}_{n_K - n_{K-1}} = S_{n_K - n_{K-1}} > \delta m, \bigcap_{i=1}^{n_K - n_{K-1}} A_i^{(n)}\Big).
\end{aligned}$$

We observe that

$$\begin{aligned}
q_{n,m} \ &\geq \ \mathbf{P}\Big(S_{am^2}^\# < (1-\delta)m, \, \overline{S}_{am^2} - S_{am^2} < \delta m, \, \overline{S}_{am^2} > \delta m\Big) + \mathbf{P}\Big(\bigcap_{i=1}^{am^2} A_i^{(n)}\Big) - 1 \\
&\geq \ \mathbf{P}\Big(S_{am^2}^\# < (1-\delta)m, \, \overline{S}_{am^2} - S_{am^2} < \delta m, \, \overline{S}_{am^2} > \delta m\Big) - \frac{\eta}{m}.
\end{aligned}$$

On the other hand, since the three events $\{S_{am^2}^\# < (1-\delta)m\}$, $\{\overline{S}_{am^2} - S_{am^2} < \delta m\}$ and $\{\overline{S}_{am^2} > \delta m\}$ are coordinately non-decreasing with respect to each $S_i - S_{i-1}$ (for $1 \leq i \leq n$), it follows from the FKG inequality that

$$q_{n,m} \geq \mathbf{P}\Big(S_{am^2}^\# < (1-\delta)m\Big) \, \mathbf{P}\Big(\overline{S}_{am^2} - S_{am^2} < \delta m\Big) \, \mathbf{P}\Big(\overline{S}_{am^2} > \delta m\Big) - \frac{\eta}{m}.$$

By Donsker's invariance principle,

$$\begin{aligned}&\lim_{m\to\infty}\mathbf{P}\Big(S^{\#}_{am^2}<(1-\delta)m\Big)\,\mathbf{P}\Big(\overline{S}_{am^2}-S_{am^2}<\delta m\Big)\,\mathbf{P}\Big(\overline{S}_{am^2}>\delta m\Big)\\ =\ &\mathbf{P}\Big(\sup_{t\in[0,\,1]}(\overline{W}(t)-W(t))<\frac{1-\delta}{\sigma\sqrt{a}}\Big)\,\mathbf{P}\Big(\overline{W}(1)-W(1)<\frac{\delta}{\sigma\sqrt{a}}\Big)\,\mathbf{P}\Big(\overline{W}(1)>\frac{\delta}{\sigma\sqrt{a}}\Big),\end{aligned}$$

where W is a standard Brownian motion, with $\overline{W}(t):=\sup_{s\in[0,\,t]}W(s)$ for any t. Recall that $\sup_{t\in[0,\,1]}(\overline{W}(t)-W(t))$ has the same distribution as $\sup_{t\in[0,\,1]}|W(t)|$; so (see Chung [18]) $\mathbf{P}\{\sup_{t\in[0,\,1]}(\overline{W}(t)-W(t))\le x\}=\exp\{-(1+o(1))\frac{\pi^2}{8x^2}\}$, for $x\to 0$. Consequently, for all sufficiently large a, say $a\ge a_0$, we have

$$\mathbf{P}\Big(\sup_{t\in[0,\,1]}(\overline{W}(t)-W(t))<\frac{1-\delta}{\sigma\sqrt{a}}\Big)\ge \mathrm{e}^{-(1+3\delta)\frac{\pi^2\sigma^2 a}{8}}.$$

Since both $\overline{W}(1)-W(1)$ and $\overline{W}(1)$ are distributed as the absolute value of a standard Gaussian random variable, we can even enlarge the value of a_0 (if necessary) such that for all $a\ge a_0$, $\mathbf{P}\{\overline{W}(1)>\frac{\delta}{\sigma\sqrt{a}}\}\ge\frac12$ and $\mathbf{P}\{\overline{W}(1)-W(1)<\frac{\delta}{\sigma\sqrt{a}}\}\ge\frac{\delta}{2\sigma\sqrt{a}}$.

Now we fix an arbitrary integer $a\ge a_0$. For all large m (say $m\ge m_0$) such that $m<\eta\sqrt{n}$, we have

$$q_{n,m}\ge \mathrm{e}^{-(1+4\delta)\frac{\pi^2\sigma^2 a}{8}}.$$

The probability $b_{n,m}$ can be estimated in a similar way: From the assumptions on $(A_i^{(n)})$ and the FKG inequality, we deduce that

$$b_{n,m}\ge\mathbf{P}(S^{\#}_{n_K-n_{K-1}}<(1-\delta)m)\,\mathbf{P}(\overline{S}_{n_K-n_{K-1}}=S_{n_K-n_{K-1}})\,\mathbf{P}(\overline{S}_{n_K-n_{K-1}}>\delta m)-\frac{\eta}{m}.$$

Observe that $am^2\le n_K-n_{K-1}\le 2am^2$. Therefore, by Donsker's invariance principle, for all $m\ge m_0$ (with an enlarged value of m_0 if necessary), $\mathbf{P}(S^{\#}_{n_K-n_{K-1}}<(1-\delta)m)\,\mathbf{P}(\overline{S}_{n_K-n_{K-1}}>\delta m)\ge c(a,\delta)$ for some constant $c(a,\delta)>0$, whereas $\mathbf{P}(\overline{S}_{n_K-n_{K-1}}=S_{n_K-n_{K-1}})=\mathbf{P}(S_1\ge 0, S_2\ge 0, \dots, S_{n_K-n_{K-1}-1}\ge 0)\ge\frac{c'}{m\sqrt{a}}$ for some constant $c'>0$. Taking $\eta:=\min\{\frac{c(a,\delta)c'}{2\sqrt{a}},\,\frac1a\}$, we get $b_{n,m}\ge\frac{\eta}{m}$. Consequently,

$$\mathbf{P}\Big(S^{\#}_n<m,\,\overline{S}_n=S_n,\,\bigcap_{i=1}^n A_i^{(n)}\Big)\ge q_{n,m}^{K-1}b_{n,m}\ge\frac{\eta}{m}\mathrm{e}^{-(1+4\delta)\frac{\pi^2\sigma^2}{8}\frac{n}{m^2}},$$

proving the lower bound in the lemma.

The upper bound is easier: with the same notation and the choice of a, we have

$$\begin{aligned}\mathbf{P}\Big(S^{\#}_n<m,\,\overline{S}_n=S_n,\,\bigcap_{i=1}^n A_i^{(n)}\Big)&\le\mathbf{P}\Big(\forall 1\le j\le K-1,\ \max_{n_{j-1}<i\le n_j}(\overline{S}_i-S_i)<m\Big)\\&\le\Big[\mathbf{P}(S^{\#}_{am^2}<m)\Big]^{K-1}.\end{aligned}$$

For all large m, $\mathbf{P}(S^{\#}_{am^2} < m) \leq \mathrm{e}^{-(1-\delta)\frac{\pi^2\sigma^2 a}{8}}$; hence $\mathbf{P}(S^{\#}_n < m, \overline{S}_n = S_n, \cap_{i=1}^n A_i^{(n)}) \leq \mathrm{e}^{-(1-\delta)\frac{\pi^2\sigma^2}{8}\frac{n}{m^2}-1}$, yielding the upper bound (by eventually modifying the choice of η in terms of a). $\square$

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We have been kindly informed by Ofer Zeitouni that Theorem 4.0.7 in the case of regular trees was independently proved by Fang and Zeitouni [32].

Chapter 5

A continuous time extension of RWRE.

The article will be organized as follows :

- In Section 2 we show Theorem 2.6.1 and 2.6.2,
- In Section 3 we show Theorem 2.6.5,
- In Section 4 we show Theorem 2.6.3 and 2.6.4.

5.1 The annealed estimate.

For any nondecreasing function $u(t)$, we will denote by $u^{-1}(t) := \inf\{v : u(v) > t\}$ the inverse function of u. We start with some preliminary statements.

5.1.1 Preliminary statements.

We first recall the Ray-Knight Theorems, they can be found in chapter XI of [82]. Let L_t^x be the local time at x before t of a brownian motion γ_t, and $\tau_t := (L_.^0)^{-1}(t)$ the inverse function of L_t^0. Let $\sigma(x)$ be the first hitting time of x by γ_t.

Statement 5.1.1 (First Ray-Knight Theorem) *The process $\{L_{\sigma(a)}^{a-t}\}_{t\geq 0}$ is a squared Bessel process, started at 0, of dimension 2 for $0 \leq t \leq a$ and of dimension 0 for $t \geq a$.*

Statement 5.1.2 (Second Ray-Knight Theorem) *Let $u \in \mathbb{R}^+$, The process $\{L_{\tau(u)}^t\}_{t\geq 0}$ is a squared Bessel process of dimension 0, starting from u.*

We have a useful representation of $H(v)$, due to Y. Hu and Z. Shi (2004). Let

$$\theta_1(v) = \int_0^{H(v)} \mathbf{1}_{\{X_s \geq 0\}} ds,$$

and

$$\theta_2(v) = \int_0^{H(v)} \mathbf{1}_{\{X_s < 0\}} ds,$$

such that $H(v) = \theta_1(v) + \theta_2(v)$.

Statement 5.1.3 *Let $\kappa \geq 0$ and $v > 0$. Under $\mathbb{P}$, we have*

$$(\theta_1(v), \theta_2(v)) \overset{\text{law}}{=} \left(4 \int_0^v \left(e^{\Xi_\kappa(s)} - 1 \right) ds, 16\Upsilon_{2-2\kappa} \left(e^{\Xi_\kappa(v)/2} \rightsquigarrow 1 \right) \right).$$

Where $\Upsilon_{2-2\kappa}(x \rightsquigarrow y)$ denotes the first hitting time of y by a Bessel process of dimension $(2-2\kappa)$ starting from x, independent of the diffusion Ξ_κ, which is the unique nonnegative solution of

$$\Xi_\kappa(t) = \int_0^t \sqrt{1 - e^{-\Xi_\kappa(s)}} d\beta'_s + \int_0^t \left(-\frac{\kappa}{2} + \frac{1+\kappa}{2} e^{-\Xi_\kappa(s)} \right) ds, \ t \geq 0. \tag{5.1.1}$$

β' being a standard brownian motion.

We shall use the following lemma from [94](Lemma 3.1).

Statement 5.1.4 *Let $\{R_t\}_{t\geq 0}$ denote a squared Bessel process of dimension 0 started at 1. For all $v, \delta > 0$, we have*

$$P\left(\sup_{0\leq t \leq v} |R_t - 1| > \delta \right) \leq 4 \frac{\sqrt{(1+\delta)v}}{\delta} \exp\left(-\frac{\delta^2}{8(1+\delta)v} \right).$$

We now turn to the proof of Theorem 2.6.2.

5.1.2 Proof of Theorem 2.6.2.

Our proof will be separated in two parts : in the first part we will deal with the positive part of $H(v)$, θ_1, then we will focus on θ_2.

The positive part.

In view of statement 5.1.3, we set

$$Z_t := e^{\Xi_\kappa(t)} - 1,$$

then Z_t is the unique nonnegative solution of

$$dZ_t = \sqrt{Z_t(1+Z_t)}d\beta_t + \left(\frac{1-\kappa}{2}Z_t + \frac{1}{2}\right)dt,$$

and

$$\theta_1(v) = 4\int_0^v Z_t dt.$$

We call

$$f(z) = \int_1^z \frac{(1+s)^\kappa}{s}ds \tag{5.1.2}$$

the scale function of Z_t.

We have

$$f(Z_t) = \int_0^t \frac{(1+Z_s)^{\kappa+\frac{1}{2}}}{\sqrt{Z_s}}d\beta_s.$$

By the Dubbins-Schwartz representation (see chapter V, Theorem (1.6) of [82]), there exists a standard Brownian motion $\gamma(t)$ such that

$$f(Z_t) = \gamma\left(\int_0^t \frac{(1+Z_s)^{2\kappa+1}}{Z_s}ds\right) := \gamma(\rho(t)). \tag{5.1.3}$$

We introduce

$$\begin{aligned}\alpha_t = \rho(t)^{-1} = \int_0^t \frac{Z_{\alpha_s}}{(1+Z_{\alpha_s})^{1+2\kappa}}ds \\ = \int_0^t \frac{f^{-1}(\gamma_s)}{[1+f^{-1}(\gamma_s)]^{1+2\kappa}}ds := \int_0^t h(\gamma_s)ds.\end{aligned} \tag{5.1.4}$$

We obtain easily the following equivalents

$$\begin{aligned}
&f(z) \sim_{z\to\infty} z^\kappa/\kappa,\\
&f(z) \sim_{z\to 0} \log z,\\
&f^{-1}(z) \sim_{z\to\infty} (\kappa z)^{1/\kappa},\\
&f^{-1}(z) \sim_{z\to-\infty} e^z,\\
&h(z) \sim_{z\to\infty} (\kappa z)^{-2},\\
&h(z) \sim_{z\to-\infty} e^z.
\end{aligned}$$

We continue with a lemma, whose proof is postponed. Let τ_t be the inverse local time of γ.

Lemma 5.1.1 *Let $\epsilon > 0$, $c_h := \int_0^\infty h(x)dx$. Let $w(t) \to \infty$, such that $w(t)/t \to 0$. Then for t large enough,*

$$\mathbb{P}\left(\rho(t) > \tau_{\frac{t}{(1-3\epsilon)c_h}}\right) \leq \exp\left(-w\right),$$

and

$$\mathbb{P}\left(\rho(t) < \tau_{\frac{t}{(1+3\epsilon)c_h}}\right) \leq \exp\left(-w\right).$$

Let $\tilde{v} \ll v$, in view of (5.1.3),

$$\theta_1(v) = 4\int_0^v f^{-1}(\gamma_{\rho(s)})ds = 4\int_0^{\rho(v)} \frac{\left(f^{-1}(\gamma_s)\right)^2}{\left[1+f^{-1}(\gamma_s)\right]^{1+2\kappa}}ds := 4\int_0^{\rho(v)} g(\gamma_s)ds. \tag{5.1.5}$$

Using lemma 5.1.1, with probability at least $1-e^{-\tilde{v}}$,

$$\int_0^{\tau\left(\frac{v}{(1+3\epsilon)c_h}\right)} g(\gamma_s)ds \leq \frac{\theta_1(v)}{4} \leq \int_0^{\tau\left(\frac{v}{(1-3\epsilon)c_h}\right)} g(\gamma_s)ds. \tag{5.1.6}$$

One can easily check that $g(x) \sim_\infty (\kappa x)^{\frac{1}{\kappa}-2}$, and $g(x) \sim_{-\infty} e^{2x}$. In view of this it is clear that the most important part of the preceding integral will come from the high values of γ_u. To be precise, for $w \in \left[\frac{v}{(1+3\epsilon)c_h}, \frac{v}{(1-3\epsilon)c_h}\right]$ and some large constant A, we have

$$\begin{aligned}\int_0^{\tau_w} g(\gamma_s)\mathbf{1}_{\gamma_s<A}ds &= \int_{-\infty}^{A} g(s)L^s_{\tau_w}ds \overset{\text{law}}{=} w^2\int_{-\infty}^{A/w} g(sw)L^s_{\tau_1}ds \\ &= w^2\int_{-\infty}^{-A\log(w)^5/w} g(sw)L^s_{\tau_1}ds + w^2\int_{-A\log(w)^5/w}^{A/w} g(sw)L^s_{\tau_1}ds := J_1+J_2. \end{aligned} \tag{5.1.7}$$

Using statement 5.1.4, for some constant $C>0$, $\mathbb{P}(J_2 > w\log(w)^5) < Ce^{-w}$. Recalling that, under the assumption of theorem 2.6.2, $v \ll \left(\frac{v}{u}\right)^{1/\kappa}$, we get that, for any $\delta > 0$, as $t \to \infty$,

$$\mathbb{P}\left[J_2 > \delta\left(\frac{v}{u}\right)^{1/\kappa}\right] \leq Ce^{-\frac{v}{(\log v)^{10}}}.$$

We postpone the proof of the following

Lemma 5.1.2 *for every $\delta > 0$, as $t \to \infty$,*

$$\mathbb{P}\left[J_1 > \delta\left(\frac{v}{u}\right)^{1/\kappa}\right] \leq Ce^{-\frac{v}{(\log v)^{10}}}.$$

As a consequence, for every $\delta > 0$, as $t \to \infty$,

$$\mathbb{P}\left[\int_0^{\tau_w} g(\gamma_s)\mathbf{1}_{\gamma_s<A}ds > 2\delta\left(\frac{v}{u}\right)^{1/\kappa}\right] \leq Ce^{-\frac{v}{(\log v)^{10}}}. \tag{5.1.8}$$

It remains to deal with $\int_0^{\tau_w} g(\gamma_s)\mathbf{1}_{\gamma_s>A}ds$. Due to the equivalent of g, for every $\epsilon > 0$, for A large enough

$$(1-\epsilon)\left(\int_0^{\tau_w}(\gamma_s)^{1/\kappa-2}\mathbf{1}_{\gamma_s>0}ds - I'\right) \leq \int_0^{\tau_w} g(\gamma_s)\mathbf{1}_{\gamma_s>A}ds$$
$$\leq (1+\epsilon)\int_0^{\tau_w}(\gamma_s)^{1/\kappa-2}\mathbf{1}_{\gamma_s>0}ds, \tag{5.1.9}$$

where

$$I' := \int_0^{\tau_w}\gamma_u^{1/\kappa-2}\mathbf{1}_{\gamma_u<A}du \overset{\text{law}}{=} w^{1/\kappa}\int_0^{A/w} y^{1/\kappa-2}L^y_{\tau(1)}dy$$

by the same computations as above. Using statement 5.1.4, for some constant $C' > 0$, with probability at least $(1-e^{-C'v})$, $L^y_{\tau(1)}$ is lesser than, say, 100 on $[0, A/w]$. Therefore

$$I' \leq 100w^{1/\kappa}\int_0^{A/w} y^{1/\kappa-2}dy < 1000A^{1/\kappa-1}w. \tag{5.1.10}$$

By the same proof as on page 218 of [49], the process

$$U_s = \int_0^{\tau_s}(\gamma_u)^{1/\kappa-2}\mathbf{1}_{\gamma_s>0}du$$

is an asymmetric κ-stable subordinator, more precisely

$$\mathbb{E}\left[\exp -\frac{\lambda}{2}U_s\right] = \exp\left(-sc_\kappa\lambda^\kappa\right),$$

where $c_\kappa = \frac{\pi}{2\kappa\sin(\pi\kappa)}\left(\frac{\kappa^\kappa}{\Gamma(\kappa)}\right)^2$. From a result of de Bruijn (see p 221 of [6]), there exists a constant C_0 such that

$$\log\mathbb{P}\left[\frac{U_s}{s^{1/\kappa}} < \left(\frac{1}{u}\right)^{1/\kappa}\right] = \log\mathbb{P}\left[U_1 < \left(\frac{1}{u}\right)^{1/\kappa}\right] \sim_\infty -C_0u^{\frac{1}{1-\kappa}}. \tag{5.1.11}$$

Similarly, by standard estimates on stable laws, for $u \to \infty$, there exists a constant C_0' such that

$$\mathbb{P}\left[\frac{U_s}{s^{1/\kappa}} > u^{1/\kappa}\right] \sim_\infty \frac{C_0'}{u}. \tag{5.1.12}$$

This, together with (5.1.6), (5.1.8), (5.1.9) and (5.1.10), implies that, for $u \to \infty$, $u \ll v^{1-\kappa}$ there exists positive constants C_1 and C_2 such that,

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}\left[\theta_1(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right]}{u^{\frac{1}{1-\kappa}}} = C_1$$

and for $u \ll e^v$,

$$\lim_{t\to\infty} u\mathbb{P}\left[\theta_1(v) > (vu)^{1/\kappa}\right] = C_2,$$

where

$$C_1 = 4^{\frac{\kappa}{1-\kappa}} \frac{C_0}{c_h^{\frac{1}{1-\kappa}}}$$

and

$$C_2 = 4^{\kappa} \frac{C_0'}{c_h}.$$

The negative part.

To finish the proof of Theorem 2.6.2, we need to deal with θ_2. Note that for $\varepsilon > 0$,

$$\mathbb{P}\left[H_v < \left(\frac{v}{u}\right)^{1/\kappa}\right] \leq \mathbb{P}\left[\theta_1(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right], \tag{5.1.13}$$

hence the lower bound in (2.6.4) is direct.

We now turn to the upper bound. We recall that u and v are two functions of t such that $u \ll v^{1-\kappa-\epsilon}$. This implies in particular that $u \ll v$. Note that

$$\mathbb{P}\left[\theta_1(v) < (1-\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa}, \theta_2(v) < \varepsilon\left(\frac{v}{u}\right)^{1/\kappa}\right] \leq \mathbb{P}\left[H(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right]. \tag{5.1.14}$$

Using statement 5.1.3, we obtain

$$\begin{aligned}\mathbb{P}&\left[\theta_1(v) < (1-\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa}, \theta_2(v) < \varepsilon\left(\frac{v}{u}\right)^{1/\kappa}\right] \\ &= \mathbb{P}\left[\Upsilon_{2-2\kappa}\left(e^{\Xi_\kappa(v)/2} \rightsquigarrow 1\right) < \varepsilon\left(\frac{v}{u}\right)^{1/\kappa}, \theta_1(v) < (1-\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa}\right].\end{aligned} \tag{5.1.15}$$

By a scaling argument, we get, for $a \geq 1$

$$\mathbb{P}\left(\Upsilon_{2-2\kappa}\left(\sqrt{a} \rightsquigarrow 1\right) < a\right) = \mathbb{P}\left(\Upsilon_{2-2\kappa}\left(1 \rightsquigarrow \frac{1}{\sqrt{a}}\right) < 1\right) \geq C > 0. \tag{5.1.16}$$

We recall from section 5.1.2 the representation

$$e^{\Xi_\kappa(v)} - 1 = f^{-1}(\gamma(\rho(t))).$$

Let $0 < \epsilon < \varepsilon/1000$, and $\delta < \varepsilon/3$ we call A the event that the condition of lemma 5.1.1 is fullfilled, that is

$$A = \left\{ \tau_{v/(1+3\epsilon)c_h} < \rho(v) < \tau_{v/(1-3\epsilon)c_h} \right\},$$

Set $\epsilon' \leq (\varepsilon\kappa)^{1/\kappa}/2$, we introduce the event

$$B := \left\{ \sup_{\tau_{v/(1+3\epsilon)c_h} < s < \tau_{v/(1-3\epsilon)c_h}} \gamma(s) < \epsilon' \frac{v}{u} \right\}.$$

Formula 4.1.2 page 185 of [14] (and the Markov property) implies

$$\mathbb{P}\left[B\right] \geq e^{-\varepsilon' u}$$

for some positive ε'. We recall from section 5.1.2 the representation

$$e^{\Xi_\kappa(v)} - 1 = f^{-1}(\gamma(\rho(t))),$$

where f^{-1} is an increasing function such that $f^{-1}(z) \sim_\infty z^\kappa/\kappa$. Therefore for t large enough, on $B \cap A$,

$$e^{\Xi_\kappa(v)} < \varepsilon \left(\frac{v}{u}\right)^{1/\kappa}.$$

Recalling equation (5.1.14), (5.1.15), and Lemma 5.1.1, we get for t large enough

$$\begin{aligned}
&\mathbb{P}\left[H(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right] \\
&\qquad \geq \mathbb{P}(B)\mathbb{P}\left[\Upsilon_{2-2\kappa}\left(e^{\Xi_\kappa(v)/2} \rightsquigarrow 1\right) < \varepsilon \left(\frac{v}{u}\right)^{1/\kappa}, \theta_1(v) < (1-\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa} |B\right] \\
&\qquad\qquad \geq \mathbb{P}(B)\mathbb{P}\left[\Upsilon_{2-2\kappa}\left(\sqrt{\varepsilon\left(\frac{v}{u}\right)^{1/\kappa}} \rightsquigarrow 1\right) < \varepsilon\left(\frac{v}{u}\right)^{1/\kappa},\right. \\
&\qquad\qquad\qquad\qquad \left.\theta_1(v) < (1-\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa} |B\right] - \mathbb{P}(B)\mathbb{P}(A^c|B).
\end{aligned}$$

Recalling lemma 5.1.1 we get

$$\mathbb{P}(B)\mathbb{P}(A^c|B) < e^{-\frac{v}{\log v}}.$$

On the other hand, $\Upsilon_{2-2\kappa}\left(\sqrt{\varepsilon\left(\frac{v}{u}\right)^{1/\kappa}} \rightsquigarrow 1\right)$ is independent of B and θ_1, and

$$\mathbb{P}\left[\Upsilon_{2-2\kappa}\left(\sqrt{\varepsilon\left(\frac{v}{u}\right)^{1/\kappa}} \rightsquigarrow 1\right) < \varepsilon\left(\frac{v}{u}\right)^{1/\kappa}\right] > C$$

by (5.1.16); therefore the upper bound in (2.6.4) will follow as soon as we show that

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}\left[\theta_1(v) < (1-\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa} |B\right]}{u^{\frac{1}{1-\kappa}}} \leq C_1 + \mu(\varepsilon),$$

where $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$. We recall from equation (5.1.5) that

$$g(x) = \frac{\left(f^{-1}(\gamma_s)\right)^2}{\left[1+f^{-1}(\gamma_s)\right]^{1+2\kappa}},$$

where f has been defined in (5.1.2).

We now recall from equation (5.1.6) that, on A

$$\frac{\theta_1(v)}{4} \leq \int_0^{\tau\left(\frac{v}{(1-3\epsilon)c_h}\right)} g(\gamma_s)\mathbb{1}_{\gamma_s>0} ds,$$

therefore

$$\begin{aligned}
&\mathbb{P}\left[\theta_1(v) < (1-\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa} |B\right] \\
&\qquad \geq \mathbb{P}\left[\int_0^{\tau\left(\frac{v}{(1-3\epsilon)c_h}\right)} g(\gamma_s) ds < (1-\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa} |B\right] \\
&\qquad\qquad - \mathbb{P}[A^c|B].
\end{aligned}$$

Once again, $\mathbb{P}[A^c|B]$ is easily bounded. On the other hand, by Ito's brownian excursion theory (see for example chapter XII of [82]), for every $l \in \mathbb{R}$, $\gamma(\tau_l + t)$ is a brownian motion started at 0, independent of $(\gamma(t))_{t\leq\tau_l}$. Therefore

$$\begin{aligned}
&\mathbb{P}\left[\int_0^{\tau\left(\frac{v}{(1-3\epsilon)c_h}\right)} g(\gamma_s) ds < (1-\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa} |B\right] \\
&\qquad \geq \mathbb{P}\left[\int_0^{\tau\left(\frac{v}{(1+3\epsilon)c_h}\right)} g(\gamma_s) ds < (1-2\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa} |B\right] \\
&\qquad\qquad \mathbb{P}\left[\int_{\tau\left(\frac{v}{(1+3\epsilon)c_h}\right)}^{\tau\left(\frac{v}{(1-3\epsilon)c_h}\right)} g(\gamma_s) ds < \varepsilon\left(\frac{v}{u}\right)^{1/\kappa} |B\right].
\end{aligned}$$

The event in the first probability on the right hand side is independent from B, therefore the conditional expectation is equal to the expectation and we can apply the results of section 5.1.2 to get

$$-\log \mathbb{P}\left[\int_0^{\tau\left(\frac{v}{(1+3\epsilon)c_h}\right)} g(\gamma_s) ds < (1-2\varepsilon)\left(\frac{v}{u}\right)^{1/\kappa} |B\right] \leq (C_1 + \mu_1(\varepsilon) + o(1)) u^{\frac{1}{1-\kappa}}.$$

On the other hand, using the Markov property,

$$\mathbb{P}\left[\int_{\tau\left(\frac{v}{(1+3\epsilon)c_h}\right)}^{\tau\left(\frac{v}{(1-3\epsilon)c_h}\right)} g(\gamma_s)ds < \varepsilon\left(\frac{v}{u}\right)^{1/\kappa} |B\right]$$
$$= \mathbb{P}\left[\int_0^{\tau_{\delta v}} g(\gamma_s)ds < \varepsilon\left(\frac{v}{u}\right)^{1/\kappa} | \sup_{0<t<\tau_{\delta v}} \gamma_s < \epsilon' \frac{v}{u}\right],$$

where $\delta = \frac{1}{((1-3\epsilon)c_h)} - \frac{1}{((1-3\epsilon)c_h)}$. Note that, as the positive and negative excursions are independent, $\int_0^{\tau_{\delta v}} g^-(\gamma_s)ds$ and B are independent, therefore we only need to bound

$$\mathbb{P}\left[\int_0^{\tau_{\delta v}} g(\gamma_s)ds < \frac{\varepsilon}{2}\left(\frac{v}{u}\right)^{1/\kappa} | \sup_{0<t<\tau_{\delta v}} \gamma_s < \epsilon' \frac{v}{u}\right]$$
$$= \mathbb{P}\left[\int_0^{\infty} g(x)L^x_{\tau_{\delta v}}dx < \frac{\varepsilon}{2}\left(\frac{v}{u}\right)^{1/\kappa} |L^\alpha_{\tau_{\delta v}} = 0\right].$$

where $\alpha = \epsilon' \frac{v}{u}$.

Intuitively, it seems clear that $\int_0^\infty g(x)L^x_{\tau_{\delta v}}dx$ will have better chances to be small if $L^\alpha_{\tau_{\delta v}} = 0$, we are going to give a rigorous proof of that.

Note that, using the second Ray-Knight theorem (Statement 5.1.2), $L^x_{\tau_{\delta v}}$ is a squared Bessel process of dimension 0 starting from δv. On the other hand, under $\mathbb{P}[\cdot|L^\alpha_{\tau_{\delta v}} = 0]$, $L^x_{\tau_{\delta v}}$ is a squared Bessel bridge of dimension 0 between δv and 0 over time α (we refer to section XI of [82] for the definition and properties of the Bessel bridge).

We are going to use Girsanov's theorem in order to compute the equation solved by the squared Bessel bridge of dimension 0. Let $\mathbf{P}_x$ and $\mathbf{P}^\alpha_{x,0}$ be respectively the distributions of the Bessel process of dimension 0 started at x and the distribution of the Bessel bridge of dimension 0 between x and 0 over time α. Let $\mathbf{E}_x$ and $\mathbf{E}^\alpha_{x,0}$ be the associated expectations. Let X_t be the canonical process and $\mathcal{F}_t$ its canonical filtration.

Using the Markov property, we get, for every $\mathcal{F}_t$-measurable function F,

$$\mathbf{E}^\alpha_{x,0}[F(X_s, s \le t)] = \frac{\mathbf{E}_x[F(X_s, s \le t), X_\alpha = 0]}{\mathbf{P}_x[X_\alpha = 0]}$$
$$= \mathbf{E}_x\left[F(X_s, s \le t)\frac{\mathbf{P}_{X_t}[X_{(\alpha-t)} = 0]}{\mathbf{P}_x[X_\alpha = 0]}\right] := \mathbf{E}_x\left[F(X_s, s \le t)h(X_t, t)\right];$$

where $h(s,t)$ can be explicitly computed (see for example Corollary XI.1.4 of [82]). We get

$$h(X_t, t) = \exp\left(\frac{x}{2\alpha} - \frac{X_t}{2(\alpha - t)}\right).$$

Using Ito's Formula, we can transform this expression to get

$$h(X_t,t) = \exp\left(-\int_0^t \frac{1}{2(\alpha-s)}dX_s + \int_0^t \frac{X_s}{2(\alpha-s)^2}ds\right).$$

Recalling that, under $\mathbf{P}_x$, X_t is a solution to

$$dX_t = 2\sqrt{X_t}d\beta_t,$$

where β is a Brownian motion, we get

$$h(X_t,t) = \exp\left(-\int_0^t \frac{\sqrt{X_s}}{(\alpha-s)}d\beta_s + \int_0^t \frac{X_s}{2(\alpha-s)^2}ds\right).$$

Therefore, thanks to Girsanov's theorem (see for example Theorem VIII.1.7 of [82]), under $\mathbf{P}^{\alpha}_{x,0}$,

$$X_t = x + \int_0^t \sqrt{X_s}d\beta_s - 2\int_0^t \frac{X_s}{(\alpha-s)}ds.$$

Coming back to our original problem, we obtain that, under $\mathbb{P}(\cdot|L^{\alpha}_{\tau_{\delta v}}=0)$, $L^x_{\tau_{\delta v}}$ is a solution to

$$X_t = \delta v + \int_0^t \sqrt{X_s}d\beta_s - 2\int_0^t \frac{X_s}{(\alpha-s)}ds.$$

while, under $\mathbb{P}$, $L^x_{\tau_{\delta v}}$ is a solution to

$$X_t = \delta v + \int_0^t \sqrt{X_s}d\beta_s.$$

Therefore, as there is pathwise uniqueness for these equation (see for example Theorem IX.3.5 of [82]), the comparison theorem (see [97]) allows us to construct a couple $(X^{(1)}, X^{(2)})$ such that $X^{(1)}$ follows the same distribution as $L^x_{\tau_{\delta v}}$ under $\mathbb{P}$, $X^{(2)}$ follows the same distribution as $L^x_{\tau_{\delta v}}$ under $\mathbb{P}(\cdot|L^{\alpha}_{\tau_{\delta v}}=0)$ and $X^{(1)} \geq X^{(2)}$ almost surely. Then one gets easily that the distribution of $\int_0^\infty g(x)L^x_{\tau_{\delta v}}dx$ under $\mathbb{P}(\cdot|L^{\alpha}_{\tau_{\delta v}}=0)$ is dominated by its distribution under $\mathbb{P}$. Then the upper bound in (2.6.4) follows easily by the results of section 5.1.2.

We now turn to the proof of (2.6.5). We have the trivial inequality

$$\begin{aligned}\mathbb{P}[\theta_1(v) > (vu)^{1/\kappa}] &\leq \mathbb{P}[\theta_1(v)+\theta_2(v) > (vu)^{1/\kappa}] \\ &\leq \mathbb{P}[\theta_1(v) > (1-\varepsilon)(vu)^{1/\kappa}] + \mathbb{P}[\theta_2(v) > \varepsilon(vu)^{1/\kappa}],\end{aligned}$$

therefore the lower bound is direct. To get the upper bound, note that $\theta_2(v)$ is increasing, so we have to show that for every $\varepsilon > 0$, and some $s > 0$,

$$\mathbb{P}\left[\Upsilon_{2-2\kappa}\left(e^{\Xi_\kappa(v+s)/2} \rightsquigarrow 1\right) > \varepsilon\,(vu)^{1/\kappa}\right] = o\left(\frac{1}{u}\right).$$

Recalling the diffusion Z_t from the last part, we need to bound

$$\mathbb{P}\left[\Upsilon_{2-2\kappa}\left(\sqrt{Z_{v+s}+1} \rightsquigarrow 1\right) > \varepsilon\,(vu)^{1/\kappa}\right] \\ = \int_0^\infty \mathbb{P}\left[\Upsilon_{2-2\kappa}\left(\sqrt{z+1} \rightsquigarrow 1\right) > \varepsilon\,(vu)^{1/\kappa}\right] d\mu_{v+s}(z), \quad (5.1.17)$$

where $\mu_v(y)$ is the distribution of Z_v. By scaling,

$$\mathbb{P}\left[\Upsilon_{2-2\kappa}\left(\sqrt{z+1} \rightsquigarrow 1\right) > \varepsilon\,(vu)^{1/\kappa}\right] \\ = \mathbb{P}\left[\Upsilon_{2-2\kappa}\left(1 \rightsquigarrow \frac{1}{\sqrt{z+1}}\right) > \varepsilon\frac{(vu)^{1/\kappa}}{z+1}\right] \leq \mathbb{P}\left[\Upsilon_{2-2\kappa}\left(1 \rightsquigarrow 0\right) > \varepsilon\frac{(vu)^{1/\kappa}}{z+1}\right].$$

It is known (see for example [98] page 40) that $\Upsilon_{2-2\kappa}(1 \rightsquigarrow 0)$ has the same distribution as $\frac{1}{2\Gamma}$ where Γ follows a distribution $\Gamma(\kappa,1)$, therefore, easy computations leads to

$$\mathbb{P}\left[\Upsilon_{2-2\kappa}\left(1 \rightsquigarrow 0\right) > \varepsilon\frac{(vu)^{1/\kappa}}{z+1}\right] \leq \left(\frac{1}{\kappa\Gamma(\kappa)(2\epsilon)^\kappa}\frac{(1+z)^\kappa}{uv}\right) \wedge 1.$$

Recalling (5.1.17), we have, for all $A > 0$

$$\mathbb{P}\left(\theta_2 > \varepsilon(uv)^{1/\kappa}\right) \leq \int_0^A \frac{1}{\kappa\Gamma(\kappa)(2\epsilon)^\kappa}\frac{(1+z)^\kappa}{uv} d\mu_{v+s}(z) + \int_A^\infty d\mu_{v+s}(z)$$

Using for example exercise VII.3.20 of [82], the diffusion Z_t has speed measure $dm(z) = \frac{2}{(1+z)^{1+\kappa}}dz$, so by Theorem 54.4 of [83] and a change of variable in order to lift the natural scale assumption, for any ϕ bounded and measurable,

$$\int_0^\infty \phi(z) d\mu_{v+s}(z) \to_{s\to\infty} \int_0^\infty \phi(z)\pi(dz),$$

with $\pi(dz) = \frac{m(dz)}{2\kappa}$. Therefore as s goes to infinity, and for some finite constant $c(\varepsilon)$,

$$\mathbb{P}\left(\theta_2 > \varepsilon(uv)^{1/\kappa}\right) \leq \frac{c(\varepsilon)}{uv}\log(1+A) + (1+A)^{-\kappa}.$$

Now, taking A such that $(1+A) \gg u^{1/\kappa}$ and $\log(1+A) \ll v$ (this is possible due to the assumptions on u and v), we get the upper bound in (2.6.5).

5.1.3 Proof of Theorem 2.6.1.

In this section we use the results for the hitting times to get the results for the diffusion itself. We begin with the proof of (2.6.2). We have the trivial inequality

$$\mathbb{P}\left(X_t > t^\kappa u\right) \leq \mathbb{P}\left[H(t^\kappa u) < t\right];$$

by taking $v = t^\kappa u$ in Theorem 2.6.2, we get the upper bound in (2.6.2). The condition $u \ll v^{1-\kappa}$ becomes $u \ll t^{1-\kappa}$.

To get the lower bound, note that, for every $\varepsilon > 0$,

$$\begin{aligned}\log \mathbb{P}\left(X_t > t^\kappa u\right) \\ &\geq \log\left[\mathbb{P}\left[H((1+\varepsilon)t^\kappa u) < t\right]\mathbb{P}\left(X_t > t^\kappa u | H((1+\varepsilon)t^\kappa u) < t\right)\right] \\ &\geq -C_1((1+\varepsilon)u)^{\frac{1}{1-\kappa}} + \log \mathbb{P}\left(X_t > t^\kappa u | H((1+\varepsilon)t^\kappa u) < t\right). \quad (5.1.18)\end{aligned}$$

The bound in the first term coming from (2.6.4). To treat the second term, note that

$$\mathbb{P}\left(X_t < t^\kappa u | H((1+\varepsilon)t^\kappa u) < t\right) \leq E\left[P_W^{(1+\varepsilon)t^\kappa u}\left[\inf_{s>0} X_s < t^\kappa u\right]\right] = \mathbb{P}\left[\inf_{s>0} X_s < -\varepsilon t^\kappa u\right],$$

by invariance of the environment.

By [52],

$$\mathbb{P}\left[\inf_{t>0} X_t < -u\right] \leq C x^{-3/2} \exp\left(-(\kappa/2)^2 x/2\right),$$

(note that c in K. Kawazu and H. Tanaka's article corresponds to $-\kappa/2$ in our setting). Therefore we get easily that, for t large enough,

$$\mathbb{P}\left(X_t < t^\kappa u | H((1+\varepsilon)t^\kappa u) < t\right) < 1/2. \quad (5.1.19)$$

The lower bound in (2.6.2) then follows from equations (5.1.18) and (5.1.19).

To prove (2.6.3), we use the fact that, for every $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\left[H\left(\frac{t^\kappa}{u} > t\right)\right] &\leq \mathbb{P}\left[X_t < \frac{t^\kappa}{u}\right] \\ &\leq \mathbb{P}\left[H\left(\frac{(1+\varepsilon)t^\kappa}{u}\right) > t\right] + \mathbb{P}\left[X_t < \frac{t^\kappa}{u}; H\left(\frac{(1+\varepsilon)t^\kappa}{u}\right) < t\right].\end{aligned}$$

Taking $v = t^\kappa/u$, Theorem 2.6.2 implies the lower bound, and the upper bound follows easily by the same argument as before.

It remains to prove Lemma 5.1.1 and Lemma 5.1.2

5.1.4 Proof of Lemmas 5.1.1 and 5.1.2.

We begin with the proof of Lemma 5.1.1. It will turn out that once the tools for this Lemma will we introduced, Lemma 5.1.2 will be quite obvious. We recall from equation (5.1.4) that

$$\alpha = \rho(t)^{-1} = \int_0^t h(\gamma_s)ds,$$

where h is some positive, integrable function. We have

$$\alpha_{\tau_t} = \int_{-\infty}^{\infty} h(x)L^x_{\tau_t}dx = tc_h + \int_{-\infty}^{\infty} h(x)\left(L^x_{\tau_t} - t\right)dx.$$

Our result then follows as soon as we show that, for t large enough

$$\mathbb{P}\left[\left|\int_{-\infty}^{\infty} h(x)\left(L^x_{\tau_t} - t\right)dx > 3t\epsilon\right|\right] < \exp\left(-w\right).$$

Let s such that $s \to \infty$ and $t/s^4 \gg w$, then

$$\begin{aligned}\int_{-\infty}^{\infty} h(x)\left(L^x_{\tau_t} - t\right)dx = \int_{-s}^{s} h(x)\left(L^x_{\tau_t} - t\right)dx \\ + \int_{s}^{\infty} h(x)\left(L^x_{\tau_t} - t\right)dx \\ + \int_{-\infty}^{s} h(x)\left(L^x_{\tau_t} - t\right)dx \\ := I_1 + I_2 + I_3.\end{aligned}$$

By a scaling argument, and using the fact that h is bounded, we have

$$|I_1| \le C\int_{-s}^{s}\left|L^x_{\tau_t} - t\right|dx \overset{\text{law}}{=} Ct\int_{-s}^{s}\left|L^{x/t}_{\tau_1} - 1\right|dx = Ct^2\int_{-s/t}^{s/t}\left|L^y_{\tau_1} - 1\right|dy.$$

Then, for t large enough,

$$\begin{aligned}\mathbb{P}(|I_1| > t\epsilon) \le \mathbb{P}\left(\sup_{y\in[-s/t,s/t]}|L^y_{\tau_1} - 1| > \frac{\epsilon}{2Cs}\right) \\ \le 2\mathbb{P}\left(\sup_{y\in[0,s/t]}|L^y_{\tau_1} - 1| > \frac{\epsilon}{2Cs}\right),\end{aligned}$$

the last bound coming from the symmetry of $L^y_{\tau_1}$ in y. On the other hand, using statement 5.1.2, $L^y_{\tau_1}$ is a squared Bessel process of dimension 0 started from 1, therefore using statement 5.1.4 with $\delta = \frac{\epsilon}{2Cs}$, $v = s/t$, we get

$$\mathbb{P}(|I_1| > t\epsilon) \leq C's \exp\left(-\frac{\epsilon^2 t}{s^3}\right) \leq \exp(-w).$$

It is clear that, for large t, $\mathbb{P}(|I_3| \geq t\epsilon) \leq \mathbb{P}(|I_2| \geq t\epsilon)$. To bound I_3, we note that, for t large enough,

$$|I_3| \leq 2\left(\int_{-\infty}^{s} \frac{L^x_{\tau_t}}{x^2}dx + t\int_{-\infty}^{s} \frac{1}{x^2}dx\right) \overset{\text{law}}{=} 2\left(\frac{t}{s} + \int_{s/t}^{\infty} \frac{L^x_{\tau_1}}{x^2}dx\right),$$

by the same scaling argument. The first part is negligible, and, using statement 5.1.2,

$$\int_{s/t}^{\infty} \frac{L^x_{\tau_1}}{x^2}dx = \int_{s/t}^{\infty} \frac{Z_x}{x^2}dx,$$

where Z_t is a squared Bessel process of dimension 0 started at 1. The following result from [81] allows us to compute the Laplace transform of this random variable.

Statement 5.1.5 (J. Pitman and M.Yor) *Let Z_t be a squared Bessel process of dimension d, starting from x, and μ a positive (Radon) measure on $(0,\infty)$ such that, for all n, $\mu(0,n) < \infty$. Then one has*

$$E\left[\exp\left(-\int Z_t d\mu(t)\right)\right] = \phi_\mu(\infty)^{d/2} \exp\left(\frac{x}{2}\phi'_\mu(0)\right),$$

where ϕ_μ is the unique decreasing and convex solution of

$$\frac{1}{2}\phi'' = \mu.\phi \text{ on } (0,\infty),\ \phi(0) = 1.$$

We note $\eta = s/t$, and $A_t = \int_\eta^\infty \frac{L^x_{\tau_1}}{x^2}dx$. The preceding statement implies that

$$\mathbb{E}\left[\exp -\lambda A_t\right] = \exp\left(\frac{1}{2}\phi'_\mu(0)\right),$$

where ϕ_μ is the solution of:

$$\phi''(x) = 2\lambda\frac{\phi(x)}{x^2}\mathbf{1}_{x\geq\eta}.$$

A decreasing solution on (η,∞) of this equation is

$$\phi(x) = C\left(\frac{x}{\eta}\right)^{\frac{1-\sqrt{1+8\lambda}}{2}}.$$

The condition $\phi(0) = 1$ and the fact that ϕ' is constant on $[0, \eta]$ implies that $C\left(1 - \frac{1-\sqrt{1+8\lambda}}{2}\right) = 1$, thus

$$\mathbb{E}\left[\exp -\lambda A_t\right] = \exp\left(\frac{1-\sqrt{1+8\lambda}}{2(1+\sqrt{1+8\lambda})\eta}\right),$$

As this function is analytic, for some $\lambda > 0$ (not depending on t),

$$\mathbb{E}\left[\exp \lambda A_t\right] = \exp\left(\frac{1-\sqrt{1-8\lambda}}{2(1+\sqrt{1-8\lambda})\eta}\right),$$

then

$$\mathbb{P}(I_2 > \epsilon t) \leq \exp\left(\frac{1-\sqrt{1-8\lambda}}{2(1+\sqrt{1-8\lambda})\eta} - \lambda\epsilon t\right),$$

from which the result follows, as $1/\eta \ll t$.

Let us now prove Lemma 5.1.2. We recall from (5.1.7) that,

$$J_1 = w^2 \int_{-\infty}^{-A\log(w)^5/w} g(sw) L^s_{\tau_1} ds \leq 2 \int_{-\infty}^{-A\log(w)^5/w} \frac{1}{s^2} L^s_{\tau_1} ds.$$

Then the proof follows easily as a corollary of the proof of Lemma 5.1.1.

5.2 Quenched slowdown.

We now turn to the proof of Theorem 2.6.5. As before we first recall some useful facts.

5.2.1 Preliminary statements.

We recall the time change representation of X_t (see, for example [46])

$$X_t = A_\kappa^{-1}\left(B(T_\kappa^{-1}(t))\right),$$

where

$$A_\kappa(x) = \int_0^x e^{W\kappa(y)} dy,$$

$$T_\kappa(t) = \int_0^t e^{-2W_\kappa(A_\kappa^{-1}(B(s)))} ds,$$

and B is a standard Brownian motion.

We also need a result about Sturm-Liouville equations. Let $V(t)$ be a positive function of $t \geq 0$, and $\bar{V}(t) = \int_0^t V(u)du$. We are interested in the solution of the differential equation

$$z''(t) = -\lambda V(t) z(t),\ t \geq 0,\ \ z(0) = 1,\ z'(0) = 0. \tag{5.2.1}$$

We have the following statement from [12] (corollary 3.2)

Statement 5.2.1 *Let $\lambda(V)$ be the supremum of all $\lambda > 0$ for which a solution to the problem (5.2.1) is positive in $[0,1)$, then*

$$\sup_{0<t<1}(1-t)\bar{V}(t) \leq \frac{1}{\lambda(V)} \leq 4 \sup_{0<t<1}(1-t)\bar{V}(t).$$

We recall the following inequality from lemma 1.1.1 of [25]

Statement 5.2.2 *Let $\gamma(t)$ be a one-dimensional brownian motion, then*

$$P\left(\sup_{0\leq s_1<s_2<t, s_2-s_1<u} |\gamma(s_2)-\gamma(s_1)| > \frac{x}{2}\right) \leq c\frac{t}{u}\exp -\frac{x^2}{9u}.$$

We finish with a useful lemma

Lemma 5.2.1 *let $a > 0$, and μ a Radon measure on $[0,a]$, and suppose there exists ϕ a positive solution of the Sturm-Liouville equation*

$$\phi'' = -\phi\mu,\ t \geq 0,\ \ \phi(a) = 1,\ \phi'(a) = 0. \tag{5.2.2}$$

Let X_t be a squared Bessel process of dimension δ, starting at x, then

$$E\left[\exp\left(\int_0^a X_t d\mu(t)\right)\right] \leq \phi(0)^{-\delta/2}\exp\left(\frac{1}{2}\frac{\phi'(0)}{\phi(0)}x\right).$$

Remark: This lemma is a extension of Statement 5.1.5, but we do not get equality in this case.

Proof: Let $F_\mu(t) = \phi'(t)/\phi(t)$, by the concavity of ϕ this is a right continuous and decreasing function, thus we can apply the integration by parts formula to get

$$F_\mu(t)X_t = F_\mu(0)x + \int_0^t F_\mu(s)dX_s + \int_0^t X_s dF_\mu(s).$$

Using (5.2.2), we can compute the last part

$$\int_0^t X_s dF_\mu(s) = \int_0^t X_s \frac{d\phi'(s)}{\phi(s)} - \int_0^t \frac{\phi'(s)d\phi(s)}{\phi(s)^2}$$
$$= -\int_0^t X_s d\mu(s) - \int_0^t X_s F_\mu(s)^2 ds.$$

Recalling that $M_t = X_t - \delta t$ is a local martingale, we set

$$Z_\mu(t) = \exp\left(\frac{1}{2}\int_0^t F_\mu(s)dM_s - \frac{1}{2}\int_0^t X_s F_\mu(s)^2 ds\right),$$

which is a positive local martingale, hence a supermartingale. Using the previous computation, we get

$$Z_\mu(t) = \exp\left(\frac{1}{2}\left[F_\mu(t)X_t - F_\mu(0)x - \delta\int_0^t F_\mu(s)ds + \int_0^t X_s d\mu(s)\right]\right).$$

As Z_μ is a supermartingale, $E[Z_\mu(a)] \leq E[Z_\mu(0)] = 1$. Therefore the result follows easily.

5.2.2 Quenched slowdown for the hitting time.

In this section we show (2.6.8). The idea of the proof is to decompose the environment in valleys of a certain size, then to study the process of the valleys visited and the time spent in the valleys. We first give a formal definition of what a valley is. For $t > 0$, $v > 0$ and $i \in \mathbb{N}$, we set $K_0 = -\lfloor t \rfloor$, and

$$K_{i+1} = \inf\left\{x > K_i, W_\kappa(K_i) - \inf_{y\in[K_i,x]} W_\kappa(y) > \frac{3}{\kappa}\log\lfloor t\rfloor,\right.$$
$$\left. W_\kappa(x) \geq \sup_{y>x} W_\kappa(y) - 1\right\}.$$

K_i is finite almost surely, due to the transience of the drifted brownian motion. The intervals $[K_i, K_{i+1}]$ will be called “valleys”. An example of such valleys is given in figure 2.

We introduce the sequence defined, for $k \geq 0$ by

$$s_0 = 0$$
$$s_{k+1} = \inf\{t > s_k, X_t \in \{K_j, j \geq 0\}\}.$$

We call $Y_k = X_{s_k}$, $l_t = \max\{i : s_i < H(v)\}$ and

$$\xi(i) = \sharp\left\{j \in [0, l_t], Y_j = K_{i+1}, Y_{j+1} = K_i\right\}.$$

We set $i_0 = \max\{j, K_j < 0\}$ and $i_1 = \max\{j, K_j < v\}$. By convention we note $K_{i_1+1} = v$. Let

$$\mathcal{B} = \sum_{i=1}^{i_1-1} \xi(i) \tag{5.2.3}$$

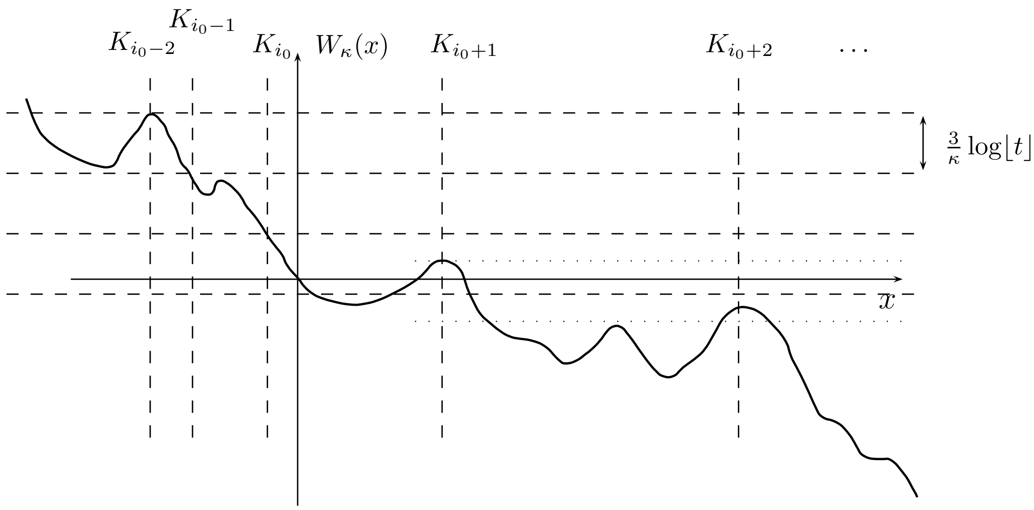

Figure 5.1: Decomposition in Valleys

denote the number of times the "walk" Y_k backtracks. Let $\theta(t)$ be the time-shift associated to the diffusion, we set for $0 \le i < i_1$

$$next(i) = \inf\{t \ge 0 : X_t = K_i, H(K_{i+1}) \circ \theta(t) < H(K_{i-1}) \circ \theta(t)\}$$

and

$$H^{next}(i) = H(K_{i+1}) \circ \theta(next(i)) - next(i).$$

We have the following decomposition of H_v :

$$H(v) = H_{init} + H_{dir} + H_{back} + H_{left} + H_{right},$$

where

$$H_{init} = \begin{cases} H(K_{i_0+1}) & \text{if } H(K_{i_0+1}) < H(K_{i_0}) \\ H(K_{i_0}) + H^{next}(i_0) \circ \theta(H(K_{i_0})) & \text{else} \end{cases}, \tag{5.2.4}$$

is the time the diffusion takes to get to K_{i_0+1},

$$H_{left} = \int_0^t \mathbf{1}_{X_t < K_1} dt, \tag{5.2.5}$$

is the time the diffusion spends at the left of K_1,

$$H_{right} = H(v) \circ \theta(next(i_1)) - next(i_1), \tag{5.2.6}$$

is the time spent to get from K_{i_1} to v

$$H_{dir} = \sum_{i=i_0+1}^{i_1-1} H^{next}(i) \tag{5.2.7}$$

is the time used for the direct crossings of the valleys and

$$H_{back} = \sum_{i=i_0+1}^{i_1-1} \sum_{j=0}^{l_t} \mathbf{1}_{Y_j=K_{i+1},Y_{j+1}=K_j} \times \left(H(K_i) \circ \theta(s_j) - s_j + H^{next}(i) \circ \theta(H(K_i) \circ \theta(s_j))\right) \tag{5.2.8}$$

is the time "lost" as a consequence of the different backtracks of Y_k.

We introduce $D_i = \sup_{K_i<s<t<K_{i+1}} W_\kappa(t) - W_\kappa(s)$, to which we will refer as the "depth" of the valley $[K_i, K_{i+1}]$, and

$$N(s,t) = \{i \geq 1, [K_i, K_i + 1) \cap [s,t) \neq \emptyset\}.$$

Note that, as seen on figure 2 there are some valleys of depth 0.

We have the following lemmas, whose proof will be postponed

Lemma 5.2.2 (environment estimates) *Let $v = t^\nu$ and $\epsilon > 0$. $\mathcal{P}$-almost surely, for $m > m_0$, for t large enough, $W \in \Omega$ where $\Omega = \Omega(t,m) = A(t) \cap G(t) \cap G(v) \cap B(t,m) \cap K(t) \cap L(t)$ and*

$$\begin{gathered}
A(t) = \left\{\max_{i\leq i_1}(K_{i+1} - K_i) \leq (\log(t))^2\right\}, \\
G(u) = \left\{\sup_{-u\leq r<s\leq u} W_\kappa(s) - W_\kappa(r) \leq \tfrac{1}{\kappa}(\log u + 3\log\log u)\right\}, \\
B(t,m) = \bigcap_{j=1}^{m-1} \left\{\sharp\{i \in N(-v,v) : D_i \geq \tfrac{1}{\kappa}\log v^{k/m} + 4\log\log(v)\} \leq v^{1-\frac{k}{m}}\right\}, \\
K(t) = \left\{\sup_{\substack{-t<t_1<t_2<t \\ |t_2-t_1|<1}} |W_\kappa(t_2) - W_\kappa(t_1)| \leq (\log t)^{1/2} \log\log t\right\}, \\
L(t) = \left\{\sup_{0<r<s<v} W_\kappa(s) - W_\kappa(r) > \tfrac{1-\epsilon}{\kappa}\log v\right\}.
\end{gathered}$$

Furthermore, whenever $u \to \infty$, the event $G(u)$ is fullfilled for u large enough.

We now turn to some quenched estimates: let $[a,c]$ be an interval of $\mathbb{R}$. We call

$$D_+ = \sup_{x\in[a,c]} \left(\max_{y\in[x,c]} W_\kappa(y) - \min_{y\in[a,x)} W_\kappa(y)\right), \tag{5.2.9}$$

$$D_- = \sup_{x\in[a,c]} \left(\max_{y\in[a,x]} W_\kappa(y) - \min_{y\in(x,c])} W_\kappa(y)\right), \tag{5.2.10}$$

and

$$D = D_- \wedge D_+.$$

We also introduce $M := \sup_{x\in[a,c]} W_\kappa(x) - \min_{x\in[a,c]} W_\kappa(x)$ We have

Lemma 5.2.3 (quenched estimates) *Let a, c, and D be as above, and $W \in \Omega$, then for some constant C, and $u > 1$*

$$\max_{x\in[a,c]} P_W^x \left[H(a) \wedge H(c) > Cu(M \vee 1)(1 \vee (c-a)^4))e^D\right] < e^{-u}. \tag{5.2.11}$$

We also have a bound on the number of backtracks. For $f \to \infty$, $f = O(t)$

$$P_W\left[\mathcal{B} \geq f\right] \leq C_3 e^{-f}. \tag{5.2.12}$$

Finally, if $W \in \Omega$, for some constant γ, for every $1 \leq i \leq i_1$, and for t large enough,

$$P_W^{K_i}\left[H(K_{i+1}) > u\gamma(\log t)^{20} e^{D_{i-1}\vee D_i} | H(K_{i+1}) < H(K_{i-1})\right] \leq e^{-u}, \tag{5.2.13}$$

$$P_W^{K_i}\left[H(K_{i-1}) > u\gamma(\log t)^{20} e^{D_{i-1}\vee D_i} | H(K_{i-1}) < H(K_{i+1})\right] \leq e^{-u}, \tag{5.2.14}$$

$$P_W^0\left[H(K_{i_0}) \wedge H(K_{i_0+1}) > u\gamma(\log t)^{20} e^{D_{i_0-1}\vee D_{i_0}}\right] \leq e^{-u}. \tag{5.2.15}$$

Thanks to these lemmas, we are able to finish the proof of Theorem 2.6.5.

Upper bound.

We recall $v = t^\nu$. Suppose $\Omega(t,m)$ is fulfilled, by the previous decomposition,

$$\begin{aligned} P_W\left(H(v) > t\right) \leq P_W\left(H_{init} > \frac{t}{5}\right) + P_W\left(H_{dir} > \frac{t}{5}\right) \\ + P_W\left(H_{back} > \frac{t}{5}\right) + P_W\left(H_{left} > \frac{t}{5}\right) + P_W\left(H_{right} > \frac{t}{5}\right). \end{aligned}$$

We begin with H_{init}. We recall from (5.2.4) that H_{init} is the time the diffusion takes to get to K_{i_0+1}. Using the precedent estimates, on $G(v)$, we have, for t large enough

$$D_{i_0} \vee D_{i_0+1} < \frac{1}{\kappa}(\log v + 3\log\log v).$$

Thus, for every $\epsilon > 0$,

$$P_W\left(H_{init} > \frac{t}{5}\right)$$
$$\leq P_W^0\left(H(K_{i_0+1}) > \frac{t^{1-\nu/\kappa}}{5}e^{D_{i_0}\vee D_{i_0+1}} \cap H(K_{i_0+1}) < H(K_{i_0})\right)$$
$$+ P_W^0\left(H(K_{i_0}) > \frac{t^{1-\nu/\kappa}}{10}e^{D_{i_0}\vee D_{i_0+1}} \cap H(K_{i_0}) < H(K_{i_0+1})\right)$$
$$+ P_W^{K_{i_0}}\left[H(K_{i_0+1}) > \frac{t^{1-\nu/\kappa}}{10}e^{D_{i_0}\vee D_{i_0+1}}|H(K_{i_0+1}) < H(K_{i_0-1})\right] \leq 3e^{-t^{1-\nu/\kappa-\epsilon}}.$$

Similarly, we have

$$P_W\left(H_{right} > \frac{t}{5}\right) = P_W^{K_{i_1}}\left(H(v) > \frac{t}{5}|H(v) < H(K_{i_1-1})\right) \leq e^{-t^{1-\nu/\kappa-\epsilon}}.$$

It is also a direct consequence of lemma 5.2.3 that, on $A(t)$, $i_0 > \frac{t}{2(\log t)^2}$, whence, recalling the definition of $\mathcal{B}$ in (3.3.16),

$$P_W\left(H_{left} > \frac{t}{5}\right) \leq P_W\left(\mathcal{B} \geq \frac{t}{4\log^2 t}\right) \leq \exp\left(-\frac{t}{4\log^2 t}\right).$$

To deal with H_{dir}, note that

$$H_{dir} = \sum_{i=i_0+1}^{i_1-1} \tau_+^{(0)}(i),$$

where $\tau_+^{(0)}(i)$ is the first crossing of the interval $[K_i, K_{i+1}]$. The $\tau_+^{(0)}(i)$ are independent random variables, and $\tau_+^{(0)}(i)$ follows the same law as $H(K_{i+1})$ under $P_W^{K_i}[\cdot|H(K_{i+1}) < H(K_{i-1})]$.

On the other hand, if $H_{dir} > t/5$, then the process spends an amount of time greater than $t/20m$ in the valleys of depth in

$$\left[\frac{k}{\kappa m}\log v + 4\log\log v, \frac{(k+1)}{\kappa m}\log v + 4\log\log v\right].$$

On $\Omega(t,m)$, the number of such valleys is at most $v^{1-\frac{k}{m}}$, we call $\sigma(k)$ the time spent in those valleys. By lemma 5.2.3, and the precedent remarks, for some constant C,

$$\frac{\sigma(k)}{C(\log t)^{11}v^{(k+1)/\kappa m}} \lhd 2v^{(1-k/m)} + \Gamma\left(2\lceil v^{(1-k/m)}\rceil, 1\right),$$

where we note $A \lhd B$ for " A is stochastically dominated by B", and $\Gamma(k, \beta)$ is the Gamma distribution of parameter (k, β).

For m large enough, one can check easily that $\nu(1-k/m) < 1-\nu(k+1)/m$ for all $k \leq m$, whence, for t large enough,

$$\begin{aligned} P_W\left[\sigma(k) \geq \frac{t}{20m}\right] &\leq P\left[\Gamma\left(2v^{(1-k/m)}, 1\right) > \frac{t^{1-\nu(k+1)/\kappa m}}{(\log t)^{12}}\right] \\ &\leq 4^{t^{\nu(1-k/m)}} \exp\left(-\frac{t^{1-\nu(k+1)/\kappa m}}{(\log t)^{12}}\right) \\ &\leq \exp\left(-2t^{1-\nu(k+2)/(\kappa m)} + \log(4) t^{\nu(1-k/m)}\right). \end{aligned}$$

Therefore, as $t \to \infty$,

$$P_W[H_{dir} > t/5] \leq m \exp\left(-t^{1-\nu(k+2)/(\kappa m)}\right) \leq m \exp\left(-t^{1-(1+\frac{2}{m})\frac{\nu}{\kappa}}\right).$$

We now deal with H_{back}.

$$\begin{aligned} &P_W\left(H_{back} > \frac{t}{5}\right) \\ &\qquad \leq \sum_{k=0}^{m-1} P_W\left(H_{back} > \frac{t}{5}, \mathcal{B} \in [t^{k/m}, t^{(k+1)/m}]\right) + P_W[\mathcal{B} > t]. \end{aligned}$$

By lemma 5.2.3, $P_W[\mathcal{B} > t] < e^{-t}$, and

$$P_W\left(H_{back} > \frac{t}{5}, \mathcal{B} \in [t^{k/m}, t^{(k+1)/m}]\right) \leq C \exp\left(-t^{k/m}\right). \tag{5.2.16}$$

On the other hand,

$$H_{back} = \sum_{i=1}^{i_1-2} \sum_{j=1}^{\xi(i)} \tau_+^{(j)}(i) + \tau_-^{(j)}(i),$$

where

- $\tau_-^{(j)}(i)$ is the $j-th$ crossing of the interval $[K_{i+1}, K_i]$.
- $\tau_+^{(j)}(i)$ is the first crossing of the interval $[K_i, K_{i+1}]$ after the $j-th$ crossing of the interval $[K_{i+1}, K_i]$.

The $\tau^{(j)}_{+,-}(i)$ are independent variables, and $\tau^{(j)}_{+}(i)$ follows the same law as $H(K_{i+1})$ under $P_W^{K_i}[\cdot|H(K_i+1) < H(K_{i-1})]$, and $\tau^{(j)}_{-}(i)$ follows the same law as $H(K_i)$ under $P_W^{K_{i+1}}[\cdot|H(K_i) < H(K_{i+2})]$, (with the convention that $K_{i_1+1} = v$). Therefore, thanks to lemma 5.2.3,

$$\frac{\tau^{(j)}_{+,-}(i)}{Ce^{H}(\log t)^{10}} \lhd 1 + \mathbf{e}$$

for some constant C and

$$H = \max_{i \in N\left(-\frac{t^{(k+1)}\log^2 t}{m}, v\right)} D_i.$$

Then, for $W_\kappa \in \Omega(n,m) \cap G\left(\frac{t^{(k+1)}\log^2 t}{m}\right)$, on the event $\{\mathcal{B} \in [t^{k/m}, t^{(k+1)/m}]\}$,

$$\frac{H_{back}}{C(t^{(k+1)/m\kappa} \vee v^{1/\kappa})(\log t)^{10}} \lhd 2t^{(k+1)/m} + \Gamma(2t^{(k+1)/m}, 1).$$

Therefore, when $1 - \frac{1}{\kappa}\left(\nu \vee \frac{k+1}{m}\right) \geq \frac{k+1}{m}$,

$$P_W\left(H_{back} > \frac{t}{5}, \mathcal{B} \in [t^{k/m}, t^{(k+1)/m}]\right) \leq C\exp\left(-C' t^{1-\frac{1}{\kappa}\left(\nu \vee \frac{k+1}{m}\right)}\right).$$

Putting this together with (5.2.16), we obtain

$$P_W\left(H_{back} > \frac{t}{5}, \mathcal{B} \in [t^{k/m}, t^{(k+1)/m}]\right) \leq C\exp\left(-C' t^{\left(1-\frac{1}{\kappa}\left(\nu \vee \frac{k+1}{m}\right)\right) \vee \frac{k}{m} - \frac{1}{m}}\right).$$

Putting together all the estimates, we get

$$\begin{aligned}\liminf_{t\to\infty} & \frac{\log(-\log P_W[H(t^\nu) > t])}{\log t} \\ &\geq \min_{k\in[-1,m+1]} \left(\frac{k}{m} \vee \left(1 - \frac{1}{\kappa}\left(\nu \vee \frac{k+1}{m}\right)\right) - \frac{1}{m}\right) \wedge \left(1 - (1 + \frac{2}{m})\frac{\nu}{\kappa}\right) \\ &\geq \left(1 - \frac{\nu}{\kappa}\right) \wedge \frac{\kappa}{\kappa+1} - \frac{3}{(1\wedge\kappa)m}, \ P-a.s..\end{aligned}$$

By taking the limit as m goes to infinity, we get the upper bound for $P_W[H(t^\nu) > t]$, namely

$$\liminf_{t\to\infty} \frac{\log(-\log P_W[H(t^\nu) > t])}{\log t} \geq \left(1 - \frac{\nu}{\kappa}\right) \wedge \frac{\kappa}{\kappa+1}.$$

We now turn to the proof of the lower bound.

Lower bound.

We suppose that $L(t)$ is fullfilled, therefore there is one valley of depth greater than $\frac{1-\epsilon}{\kappa}\log v$ before v. Let b be the bottom of this valley, and c such that $b < c$ and

$$W_\kappa(c) - W_\kappa(b) = \frac{1-\epsilon}{\kappa}\log v.$$

It is easy to see that $H(v) \geq H(c) - H(b)$, whence

$$P_W[H(t^\nu) > t] \geq P_W^b[H(c) > t].$$

We can suppose, without loss of generality, that $b = 0$. By the time change representation from the preliminary statements, under P_W, $H(c) = T_\kappa(\sigma(A_\kappa(c)))$, where $\sigma(x)$ is the first hitting time of x by a brownian motion B. Therefore

$$\begin{aligned} H(c) = \int_0^{\sigma(A_\kappa(c))} e^{-2W_\kappa(A_\kappa^{-1}(B_s))} ds. \\ = \int_{-\infty}^{A_\kappa(c)} \exp\left(-2W_\kappa(A_\kappa^{-1}(x))\right) L^x_{\sigma(A_\kappa(c))} dx \\ = \int_{-\infty}^{c} \exp\left(-W_\kappa(u)\right) L^{A_\kappa(u)}_{\sigma(A_\kappa(c))} du. \end{aligned}$$

The last equality coming from a change of variable in the integral. By a scaling argument, we get

$$H(c) \overset{\text{law}}{=} \int_{-\infty}^{c} \exp\left(-W_\kappa(u)\right) A_\kappa(c) L^{A_\kappa(u)/A_\kappa(c)}_{\sigma(1)} du.$$

We suppose $W_\kappa \in K(t)$, so

$$A_\kappa(c) \geq e^{W_\kappa(c) - (\log t)^{2/3}} > t^{(1-2\epsilon)\frac{\nu}{\kappa}},$$

and $A_\kappa(-1) > -e^{-(\log t)^{2/3}}$. Hence

$$H(c) \rhd t^{(1-3\epsilon)\frac{\nu}{\kappa}} \inf_{x \in [A_\kappa(-1)/A_\kappa(c), 0]} L^x_{\sigma(1)}.$$

For t large enough, $A_\kappa(-1)/A_\kappa(c) > -1/2$. Therefore by the first Ray-Knight theorem (Statement 5.1.1)

$$\begin{aligned} P_W^b[H(c) > t] \geq P\left[\inf_{x \in [-1/2, 0]} L^x_{\sigma(1)} > t^{1-\frac{\nu}{\kappa}+\varepsilon}\right] \\ \geq P\left[Z_1' > 2t^{1-\frac{\nu}{\kappa}+\varepsilon}\right] P\left[\sup_{u \in [0,1/2]} |Z_u| < t^{1-\frac{\nu}{\kappa}+\varepsilon}\right], \end{aligned}$$

where Z_t is a squared Bessel process of dimension 0 started at 0 and Z'_t is a squared Bessel process of dimension 2 started at 0. The last probability is greater than 1/2 for t large enough, and the first one is explicitly known (see for example [14]). We obtain that, for all $\varepsilon > 0$,

$$P_W[H(v) > t] \geq \exp\left(-\frac{1}{2} t^{1-\frac{\nu}{\kappa}+\varepsilon}\right).$$

To obtain the other lower bound, note that, similarly to lemma 5.2.2, almost surely, there is a valley of depth at least $\frac{1-\epsilon}{\kappa+1} \log t$ in $[-t^{\kappa/(\kappa+1)}, 0]$, let b' be the bottom of such valley, and $c' > b'$ such that

$$W_\kappa(c') - W_\kappa(b') \geq \frac{1-\epsilon}{\kappa+1} \log t.$$

We have

$$P_W[H(v) > t] \geq P_W[H(b) < H(t^\nu)] P_W^b[H(c) > t].$$

Recalling the time change representation,

$$P_W[H(b) < H(t^\nu)] = \frac{A_\kappa(t^\nu)}{A_\kappa(t^\nu) - A_\kappa(b)}.$$

when $W_\kappa \in K(t)$, we can easily show that for every $\epsilon > 0$, as n goes to infinity,

$$P_W[H(b) < H(t^\nu)] \geq \exp -t^{\frac{\kappa}{\kappa+1}+\epsilon}.$$

By the same computations as for the first bound, we get

$$P_W^b[H(c) > t] \geq \exp -t^{\frac{\kappa}{\kappa+1}+\epsilon}.$$

Putting together both inequalities, we get

$$\liminf_{t\to\infty} \frac{\log(-\log P_W[H(t^\nu) > t])}{\log t} \leq \left(1 - \frac{\nu}{\kappa}\right) \wedge \frac{\kappa}{\kappa+1},$$

which finishes the proof of Theorem 2.6.5.

5.2.3 Quenched slowdown for the diffusion.

In this section we finish the proof of Theorem 2.6.5. The lower bound is trivial, since

$$P_W[X_t < t^\nu] \geq P_W[H(t^\nu) > t].$$

To get the upper bound, let $m \in \mathbb{N}$, note that

$$\begin{aligned} P_W[X_t < t^\nu] \leq P_W[H(t^\nu) > t] \\ + \sum_{k=0}^{m-1} P_W\left[H\left(t^{\nu+\frac{k}{m}}\right) < t < H\left(t^{\nu+\frac{k+1}{m}}\right)\right] P_W^{t^{\nu+\frac{k}{m}}}[H(t^\nu) < t] \\ + P_W[H(t^{\nu+1}) < t] P_W^{t^{\nu+1}}[H(t^\nu) < t]. \end{aligned} \tag{5.2.17}$$

Using the explicit distribution of the supremum before t of a drifted brownian motion (see page 197 of [14]) and the Borel-Cantelli lemma, we can easily see that for every $k \in \{1, m\}$, the event

$$U_m^k(n) := \left\{ \sup_{(n+1)^\nu < s < t < n^{\nu+\frac{k}{m}}} W_\kappa(s) - W_\kappa(t) \geq \frac{\kappa}{4} n^{\nu+\frac{k}{m}} \right\}$$

is fullfilled for all n large enough, therefore so does

$$U_n = \bigcup_{k=1}^m U_m^k(n).$$

Hence on $U_{\lceil t \rceil}$, there exist $t^\nu < a < b < t^{\nu+\frac{k}{m}}$ such that

$$W_\kappa(a) - W_\kappa(b) \geq \frac{\kappa}{4} t^{\nu+\frac{k}{m}}.$$

By the same computations as in part 5.2.2, we get that, on $U(\lceil t \rceil)$,

$$\begin{aligned} P_W^b[H(a) < t] &\leq P_W\left[e^{\frac{\kappa}{8} t^{\nu+\frac{k}{m}}} \inf_{x \in [0, e^{-\frac{\kappa}{8} t^{\nu+\frac{k}{m}}}]} L_{\sigma(1)}^x < t \right] \\ &\leq P_W\left[\inf_{u \in [1, 1-e^{-\frac{\kappa}{8} t^{\nu+\frac{k}{m}}}]} Z_u < t e^{-\frac{\kappa}{8} t^{\nu+\frac{k}{m}}} \right], \end{aligned}$$

where Z_u is a squared Bessel process of dimension 2 started at zero. We have

$$\begin{aligned} P_W &\left[\inf_{u \in [1, 1-e^{-\frac{\kappa}{8} t^\alpha}]} Z_u < t e^{-\frac{\kappa}{8} t^\alpha} \right] \\ &\leq P_W\left[Z_1 < 2t e^{-\frac{\kappa}{8} t^\alpha} \right] + P_W\left[\sup_{u \in [1, 1-e^{-\frac{\kappa}{8} t^\alpha}]} |Z_u - Z_1| \geq t e^{-\frac{\kappa}{8} t^\alpha} \right]. \end{aligned}$$

Using statement 5.2.2 with $u = te^{-\frac{\kappa}{8}t^\alpha}$ and the fact that $\sqrt{Z_{1-t} - Z_1}$ is the Euclidean norm of a two dimensional Brownian motion, we get

$$P_W\left[\sup_{t\in[1,1-e^{-\frac{\kappa}{8}t^\alpha}]} |Z_t - Z_1| \geq te^{-\frac{\kappa}{8}t^\alpha}\right] \leq 2\exp -\frac{t}{10}.$$

On the other hand, by the exact distribution of Z_1,

$$P(Z_1 < x) = 1 - e^{-x/2} < x.$$

Therefore we get that for some constant C

$$P_W^{t^{\nu+\frac{k}{m}}}[H(t^\nu) < t] \leq P_W^b[H(a) < t] < e^{-Ct^{\nu+\frac{k}{m}}}.$$

On the other hand, the bound for the hitting time implies that

$$P_W\left[H\left(t^{\nu+\frac{k}{m}}\right) < t < H\left(t^{\nu+\frac{k+1}{m}}\right)\right] \leq \exp\left(-t^{\left(1-\left(\nu+\frac{k+1}{m}\right)/\kappa\right)\wedge\left(\frac{\kappa}{\kappa+1}\right)-\frac{1}{m}}\right),$$

indeed the bound is trivial when $\nu + k/m > \kappa$.

The same arguments apply to the other terms of (5.2.17), whence

$$\liminf_{t\to\infty} \frac{\log(-\log P_W[X_t < t^\nu])}{\log t} \geq \min_{k\in[0,m]}\left[\left(\nu + \frac{k}{m}\right) \vee \left(\left(1 - \frac{\nu + (k+1)/m}{\kappa}\right) \wedge \frac{\kappa}{\kappa+1} - \frac{1}{m}\right)\right].$$

Minimizing over k and taking the limit as m go to infinity, we get the desired upper bound.

5.2.4 Proof of the lemmas.

We begin with the estimates on the environment.

Proof of lemma 5.2.2.

Note that, as an easy consequence of statement 5.2.2, almost surely for t large enough $i_1 < 2t$. Therefore

$$A(t) \supset \tilde{A}(\lfloor t\rfloor) := \left\{\max_{i\leq 2\lfloor t\rfloor+1} |K_{i+1} - K_i| \leq \log^2(\lfloor t\rfloor)\right\}. \tag{5.2.18}$$

Let us show that

$$\mathcal{P}[\tilde{A}(n)^c] = O(1/n^2). \tag{5.2.19}$$

We have

$$\mathcal{P}[\tilde{A}(n)^c] \le \sum_{i=0}^{2n+1} \mathcal{P}[K_{i+1} - K_i \ge (\log(n))^2]. \tag{5.2.20}$$

By invariance of the environment,

$$\mathcal{P}[K_1 - K_0 \ge (\log(n))^2] = \mathcal{P}[\tilde{K}_1 \ge (\log(n))^2],$$

where

$$\tilde{K}_1 = \min\left\{t \ge 0 : -\min_{s\in[0,n]} W_\kappa(s) \ge \frac{3}{\kappa}\log n, W_\kappa(t) > \sup_{s\ge t} W_\kappa(s) - 1\right\}.$$

On the other hand, conditionally to K_i, the process $W_\kappa(K_i+s) - W_\kappa(K_i)$ is a drifted Brownian motion conditionned to have its supremum lesser than 1. Therefore

$$\mathcal{P}[K_{i+1} - K_i \ge (\log n)^2] = \mathcal{P}[\tilde{K}_1 \ge (\log n)^2 | \sup_{t\ge 0} W_\kappa \le 1]$$
$$\le \frac{\mathcal{P}[\tilde{K}_1 \ge (\log n)^2]}{\mathcal{P}[\sup_{t\ge 0} W_\kappa \le 1]}.$$

For $\kappa > 0$, $\mathcal{P}[\sup_{t\ge 0} W_\kappa \le 1]$ is a positive constant. It remains to bound $\mathcal{P}[\tilde{K}_1 \ge (\log n)^2]$, note that if

$$W_\kappa\left((\log n)^2\right) < -\frac{6}{\kappa}\log n,$$

and

$$\sup_{t\ge(\log n)^2} W_\kappa(t) - W_\kappa\left((\log n)^2\right) < \frac{3}{\kappa}\log n,$$

then there exists one point x^* before $(\log n)^2$ such that $\inf_{t\in[0,x^*]} W_\kappa(t) < -\frac{3}{\kappa}\log n$ and $W_\kappa(x^*) \ge \sup_{s\ge x^*} W_\kappa(s) - 1$ (see figure 3), therefore $\tilde{K}_1 < (\log n)^2$. Taking the complementary events, we get

$$\mathcal{P}[\tilde{K}_1 \ge (\log n)^2]$$
$$\le \mathcal{P}\left[W_\kappa\left((\log n)^2\right) > -\frac{6}{\kappa}\log n \text{ or } \sup_{t\ge(\log n)^2} W_\kappa(t) - W_\kappa\left((\log n)^2\right) > \frac{3}{\kappa}\log n\right].$$

By standard gaussian estimates,

$$\mathcal{P}\left[W_\kappa\left((\log n)^2\right) > -\frac{6}{\kappa}\log n\right] = O(n^{-3})$$

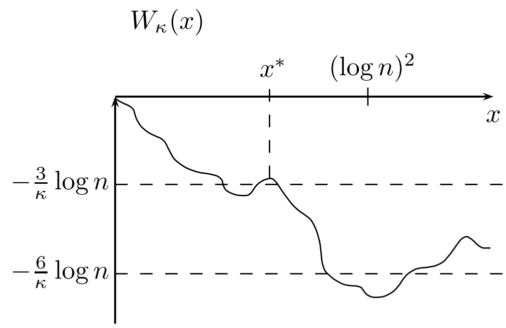

Figure 5.2: $\tilde{K}_1$

and

$$\mathcal{P}\left[\sup_{t\geq(\log n)^2} W_\kappa(t) - W_\kappa\left((\log n)^2\right) > \frac{3}{\kappa}\log n\right] = \mathcal{P}\left[\sup_{t\geq 0} W_\kappa(t) > \frac{3}{\kappa}\log n\right].$$

By formula 1.1.4(1) from page 197 of [14], the last probability is equal to n^{-3}. Therefore recalling equation (5.2.20), this finishes the proof of (5.2.19). Therefore, using the Borel-Cantelli lemma and (5.2.18), $A(t)$ is fullfilled for every t large enough.

We now turn to G. We consider the process

$$U_t := \sup_{-\infty\leq s\leq t} W_\kappa(t) - W_\kappa(s). \tag{5.2.21}$$

Note that for $n = \lfloor t \rfloor$,

$$\left\{\sup_{-(n+1)\leq t\leq n+1} U_t \leq \frac{1}{\kappa}(\log n + 3\log\log n)\right\} \subset G(t).$$

The process U_t is called a Reflected Brownian Motion with drift. This kind of process appears naturally in some queueing system models. It is a positive and stationnary diffusion, with stationnary law the exponential law of parameter κ. It is also reversible in time, therefore we can reduce to proving that, as n goes to infinity, the event

$$\left\{\sup_{0\leq t\leq n+1} U_t \leq \frac{1}{\kappa}(\log n + 3\log\log n)\right\} \tag{5.2.22}$$

is fullfilled.

In [86] it is shown that the length of the excursions away from zero (or busy periods) of U_t follows a gamma distribution $\Gamma\left(\frac{1}{2}, \frac{\kappa^2}{8}\right)$, and that the supremum m_0 over one excursion of U_t has an explicit law, given by

$$\mathcal{P}(m_0 > y) = \frac{2e^{-\kappa y}}{(1-e^{-\kappa y})^2}(\kappa y - (1 - e^{-\kappa y})). \tag{5.2.23}$$

Let C be some large constant. We call $F(n)$ the event that U_t makes more than Cn excursions between time 0 and time $n+1$. We have

$$\mathcal{P}(F(n)) \leq \mathcal{P}\left(\Gamma\left(\frac{Cn}{2}, \frac{\kappa^2}{8}\right) < n+1\right) = \frac{\gamma(Cn/2, \frac{(n+1)\kappa^2}{8})}{\Gamma(Cn/2)},$$

where $\gamma(\cdot,\cdot)$ is the incomplete gamma function. By Stirling's formula,

$$\mathcal{P}(F(n)) = O(((n+1)\kappa^2/8)^{Cn/2}(Cn/2e)^{-Cn/2-1/2}) = o(n^{-4})$$

for C large enough. Therefore by the Borel-Cantelli lemma, almost surely there exists n_0 such that $F(n)$ is fullfilled for all $n \geq n_0$.

On the other hand, we call $\tilde{G}(k)$ the event that the maximum during the $k-th$ excursion is lower than $1/\kappa(\log k + 3\log\log k)$. Recalling (5.2.23), for $k \geq 10$,

$$\mathcal{P}\left(\tilde{G}(k)^c\right) = \mathcal{P}\left(m_0 > \frac{1}{\kappa}\left(\log k + 3\log\log k\right)\right) \leq \frac{8}{k(\log k)^2}.$$

By the Borel-Cantelli lemma, we get that there exists k_0 such that $\tilde{G}(k)$ is fullfilled for all $k \geq k_0$. Take $n > n_0 \vee k_0$, and such that

$$\frac{1}{\kappa}(\log n + 3\log\log n)$$

is greater than the supremum over the k_0 first excursions of U_t. Then on $F(n) \cap \bigcap_{k=k_0}^{n} \tilde{G}(k)$ the event in (5.2.22) is fullfilled. This implies the result for $G(t)$.

Let us turn to $B(t,m)$. Let $n = \lfloor v \rfloor$. We call, for $0 < a < 1$

$$\tilde{B}(n,a) = \left\{\sharp\left\{i \in N[-(n+1), n+1] : D_i \geq \frac{a}{\kappa}\log n + \frac{4}{\kappa}\log\log n\right\} < n^{1-a}\right\}.$$

Recalling the definitions of the K_i and U_t, we note that the event that two different K_i belong to the same excursion of U_t implies that the maximum during this excursion is at least $3/\kappa \log n$, therefore, by the same argument as before, when n is large enough, this does not happen. We can also suppose that U_t makes less than Cn excursions between time $-(n+1)$ and $n+1$. Thus, on these events,

$$\sharp\left\{i \in N[-(n+1), (n+1)] : H_i \geq \frac{a}{\kappa}\log n + 4\log\log n\right\}$$

is stochastically dominated by a $Binomial(2n+1, p)$, where

$$p = P\left[m_t \geq \frac{a}{\kappa}\log n + 4\log\log n\right] < 2\frac{n^{-a}}{\log n^2}.$$

Whence, using Chebyshev's exponential inequality,

$$P[\tilde{B}(n,a)^c] \leq \exp\left(-n^{1-a}\right) \exp\left((2n+1)\log(1+p(e-1))\right) \leq \exp\left(4np - n^{1-a}\right).$$

The estimate on p, together with the Borel-Cantelli lemma, implies that, almost surely for n large enough,

$$\bigcap_{1}^{m-1} \tilde{B}(n,k/m) \subset B(t,m)$$

is fullfilled.

We finally prove that $L(t)$ is fullfilled for t large enough. Recalling the notations concerning U_t from (5.2.21), we call $f(n)$ the event that U_t makes more that $\frac{n}{(\log n)^2}$ excursions before time n. Using the explicit distribution of the length of the excursions of U_t, we have

$$\mathcal{P}\left(f(n)^c\right) \leq \mathcal{P}\left(\Gamma\left(\frac{n}{2(\log n)^2}, \frac{\kappa^2}{8}\right) > n\right).$$

Recalling that a $\Gamma(k,\theta)$ distribution has expectation $k\theta$ and variance $k\theta^2$, by Bienaymé-Chebyshev's inequality, for n large,

$$\mathcal{P}\left(f(n)^c\right) \leq \frac{10}{n(\log n)^2}.$$

Now the Borel-Cantelli lemma implies that $f(n)$ is fullfilled for all n large enough.

Now suppose that $f(\lfloor v \rfloor)$ is fullfilled, Note that U_t and $\sup_{0<s<t} W_\kappa(t) - W_\kappa(s)$ are equal after the first 0 of U_t. Call $\tilde{L}(t)$ the event that there exists one excursion of height at least $\frac{1-\epsilon}{\kappa}\log(n+1)$ between the second and the $\lfloor \frac{n}{(\log n)^2} \rfloor$-th excursion of U_t. It is easy to see that

$$f(\lfloor v \rfloor) \cap \tilde{L}(t) \subset L(t).$$

On the other hand, by (5.2.23),

$$\mathcal{P}\left(\tilde{L}(t)^c\right) \leq \mathcal{P}\left[m_t < \frac{1-\epsilon}{\kappa}\log(n+1)\right]^{\frac{n}{(\log n)^2}}$$
$$\leq \left(1 - e^{-(1-\epsilon)\log n+1}\right)^{\frac{n}{(\log n)^2}}.$$

This is summable, therefore we can apply the Borel-Cantelli lemma to get the result on $\tilde{L}(t)$, then on $L(t)$.

The result on $K(t)$ is a direct consequence of statement 5.2.2.

We now turn to the quenched estimates.

Proof of Lemma 5.2.3.

We begin with the proof of (5.2.11). Without loss of generality we can suppose $x = 0$ and $D = D_+$. We suppose $|c - a| \geq 1$, the proof being similar when $|c - a| \leq 1$.

Recalling from the preliminary statements the time change representation of X_t, we get that, under P_W, $H(v) = T_\kappa(\sigma(A_\kappa(v)))$, where

$$A_\kappa(x) = \int_0^x e^{W\kappa(y)} dy,$$

$$T_\kappa(t) = \int_0^t e^{-2W_\kappa(A_\kappa^{-1}(B(s)))} ds,$$

and $\sigma(x)$ is the first hitting time of x by a Brownian motion B. Therefore

$$\begin{aligned} H(a) \wedge H(c) = \int_0^{\sigma(A_\kappa(a)) \wedge \sigma(A_\kappa(c))} & e^{-2W_\kappa(A_\kappa^{-1}(B_s))} ds \\ &= \int_{A_\kappa(a)}^{A_\kappa(c)} \exp\left(-2W_\kappa(A_\kappa^{-1}(x))\right) L^x_{\sigma(A_\kappa(a)) \wedge \sigma(A_\kappa(c))} dx. \end{aligned}$$

We are going to use the second Ray-Knight Theorem (Statement 5.1.2) : note that

$$L^x_{\sigma(A_\kappa(a)) \wedge \sigma(A_\kappa(c))} \leq L^x_{\sigma(A_\kappa(c))},$$

and that $L^x_{\sigma(A_\kappa(c))}$ is stochastically dominated by the local time at x before $\sigma(A_\kappa(c))$ of a Brownian motion started at a. Therefore

$$H(a) \wedge H(b) \lhd \int_0^{A_\kappa(c) - A_\kappa(a)} V(s) X_s ds,$$

where $V(x) = \exp\left(-2W_\kappa(A_\kappa^{-1}(A_\kappa(c) - x))\right)$, and X_s is a Bessel process of dimension 2, started at 0. We call $\alpha := A_\kappa(c) - A_\kappa(a)$, and $\lambda(V)$ the supremum of all λ such that a solution to

$$y''(t) = -\lambda V(t) y(t),\ t \geq 0\ y'(\alpha) = 0,\ y(\alpha) = 1$$

is positive in $[0, \alpha]$. $\lambda(V)$ is usually known as the spectral gap, or Poincaré's constant associated to V.

By a standard change of variable in the previous differential equation, and an application of Statement 5.2.1, we get

$$\frac{1}{\lambda(V)} \leq 32(A_\kappa(c)-A_\kappa(a))^2 \sup_{0<t<1}(1-t)\int_0^t e^{-2W_\kappa(A_\kappa^{-1}(A_\kappa(a)+s(A_\kappa(c)-A_\kappa(a))))}ds$$

$$= 32(A_\kappa(c)-A_\kappa(a)) \sup_{0<t<1}(1-t)\int_{A_\kappa(a)}^{A_\kappa(a)+t(A_\kappa(c)-A_\kappa(a))} e^{-2W_\kappa(A_\kappa^{-1}(u))}du$$

$$= 32(A_\kappa(c)-A_\kappa(a)) \sup_{0<t<1}(1-t)\int_a^{d(t)} e^{-W_\kappa(v)}dv,$$

where $d(t) = A_\kappa^{-1}(A_\kappa(a)+t(A_\kappa(c)-A_\kappa(a)))$. Easy computations show that

$$(1-t)(A_\kappa(c)-A_\kappa(a)) = \int_{d(t)}^c e^{W_\kappa(v)}dv,$$

whence, recalling from (5.2.9) that

$$D_+ = \sup_{x\in[a,c]}\left(\max_{y\in[x,c]} W_\kappa(y) - \min_{y\in[a,x)} W_\kappa(y)\right),$$

we get

$$\frac{1}{\lambda(V)} \leq 32 \sup_{a\leq x\leq c}\int_a^x e^{-W_\kappa(v)}dv\int_x^c e^{W_\kappa(v)}dv \leq 32(c-a)e^{D^+}.$$

From Lemma 5.2.1 we get that $E[\exp\lambda(V)U]$ is finite, but we need an explicit bound. Toward this goal we are going to extend the interval : let c' be such that $(c'-a)=2(c-a)$ and let us extend W_κ on $[c,c']$ by a constant function (equal to $W_\kappa(c)$). We call $\tilde{V}(x) = \exp\left(-2W_\kappa(A_\kappa^{-1}(A_\kappa(c)-x))\right)$, for $x\in[A_\kappa(c)-A_\kappa(c'), A_\kappa(c)-A_\kappa(a)]$ and $\lambda(\tilde{V})$ the supremum of all λ such that a solution to

$$y''(t) = -\lambda\tilde{V}(t)y(t),\ t\geq\ y'(\alpha)=0,\ y(\alpha)=1 \tag{5.2.24}$$

is positive in $[A_\kappa(c)-A_\kappa(c'),\alpha]$.

By the same calculations as before we get

$$\frac{1}{\lambda(\tilde{V})} \leq 32 \sup_{a\leq x\leq c'}\int_a^x e^{-W_\kappa(v)}dv\int_x^{c'} e^{W_\kappa(v)}dv \leq 32(c'-a)e^{D^+}$$
$$= 64(c-a)e^{D^+}.$$

For $\lambda<\lambda(\tilde{V})$, let ϕ be a solution to (5.2.24) on $[A_\kappa(c)-A_\kappa(c'),\alpha]$, then ϕ is a solution to (5.2.24) on $[0,\alpha]$, and by concavity,

$$\phi(0) \geq \frac{A_\kappa(c')-A_\kappa(c)}{A_\kappa(c')-A_\kappa(a)} \geq \frac{e^{-M}}{2}.$$

Together with lemma 5.2.1, we get

$$E_W[\exp(\lambda H(a) \wedge H(c))] < 2e^M.$$

This, together with Markov's inequality, finishes the proof of the first part of lemma 5.2.3.

In order to prove (5.2.12), note that, due to the time change representation, and for $W \in \Omega$,

$$\begin{aligned} P_W^{K_i}\left[H(K_{i-1}) < H(K_{i+1})\right] &= \int_{K_i}^{K_{i+1}} e^{W_\kappa(x)} dx \left(\int_{K_{i-1}}^{K_{i+1}} e^{W_\kappa(x)} dx \right)^{-1} \\ &\leq \max_{i \leq i_1} (K_i - K_{i-1}) \frac{e^{1+W_\kappa(K_i)}}{e^{W_\kappa(K_{i-1}) - (\log t)^{1/2} \log\log t}} \leq t^{-3/2}, \quad (5.2.25) \end{aligned}$$

using the fact, that, by definition of the K_i, on $K(t) \cap G(t)$,

$$W_\kappa(K_{i-1}) \geq \inf_{K_{i-1} \leq x \leq K_i} W_\kappa(x) + \frac{3}{\kappa} \log t \geq W_\kappa(K_i) + \frac{2}{\kappa} \log t - \frac{3}{\kappa} \log\log t.$$

Then we have to distinguish two cases : either the walk Y_j gets to the level v in more than $3n$ steps or in less than $3n$ steps. In the first case there are at least n steps back before $H(v)$, and in the second case the number of steps back is dominated by a $Binomial(3n, n^{-3/2})$. Thus

$$P_W\left[\mathcal{B} \geq f(t)\right] \leq \begin{pmatrix} 3n \\ n \end{pmatrix} \left(\frac{1}{n^{3/2}} \right)^n + P\left[Binomial(3n, n^{-3/2}) \geq f(t)\right].$$

The result follows easily from Stirling's formula and Chebyshev's exponential inequality.

We now turn to the proof of (5.2.13),(5.2.14) and (5.2.15). We start with (5.2.13). First note that

$$\begin{aligned} &P_W^{K_i}\left[H(K_{i+1}) > u\gamma(\log t)^{20} e^{D_{i-1} \vee D_i} | H(K_{i+1}) < H(K_{i-1})\right] \\ &\qquad \leq \frac{P_W^{K_i}\left[H(K_{i-1}) \wedge H(K_{i+1}) > u\gamma(\log t)^{20} e^{D_{i-1} \vee D_i})\right]}{P_W^{K_i}\left[H(K_{i+1}) < H(K_{i-1})\right]}, \end{aligned}$$

As a direct consequence of (5.2.25), we have, $P - a.s.$, for n large enough,

$$P_W^{K_i}\left[H(K_{i+1}) < H(K_{i-1})\right] \geq \frac{1}{2}.$$

We are going to use (5.2.11) in order to bound the numerator. Note that, due to the definition of the K_i,

$$\sup_{K_{i-1} < s < t < K_{i+1}} W_\kappa(s) - W_\kappa(t) \geq D_{i-1} \vee D_i.$$

On the other hand, on $A(t) \cap K(t)$,

$$K_{i+1} - K_{i-1} \leq 2(\log t)^2,$$

and then

$$\sup_{x \in [K_{i-1}, K_{i+1}]} W_\kappa(x) - \min_{x \in [K_{i-1}, K_{i+1}]} W_\kappa(x) < (\log t)^3.$$

Therefore, the result follows easily by application of (5.2.11).

We now turn to the proof of (5.2.14). As before,

$$P_W^{K_i}\left[H(K_{i-1}) > u\gamma(\log t)^{20} e^{D_{i-1} \vee D_i} | H(K_{i-1}) < H(K_{i+1})\right] \leq \frac{P_W^{K_i}\left[H(K_{i-1}) \wedge H(K_{i+1}) > u\gamma(\log t)^{20} e^{D_{i-1} \vee D_i}\right]}{P_W^{K_i}\left[H(K_{i-1}) < H(K_{i+1})\right]}.$$

The numerator is the same as in the proof of (5.2.13), so we only have to deal with the denominator. We recall from (5.2.25) that

$$P_W^{K_i}\left[H(K_{i-1}) < H(K_{i+1})\right] = \int_{K_i}^{K_{i+1}} e^{W_\kappa(x)} dx \left(\int_{K_{i-1}}^{K_{i+1}} e^{W_\kappa(x)} dx\right)^{-1}.$$

On $K(t) \cap G(t)$, we obtain easily

$$P_W^{K_i}\left[H(K_{i-1}) < H(K_{i+1})\right] \geq \frac{e^{W_\kappa(K_i) - W_\kappa(K_{i-1}) - \log t}}{(\log t)^3}.$$

Note that on $A(t) \cap K(t)$, $W_\kappa(K_{i-1}) - W_\kappa(K_i) \leq (\log t)^3$. (5.2.14) follows then easily.

The proof of (5.2.15) is similar and omitted.

5.3 Quenched speedup.

In this part we show Theorem 2.6.3. We first recall some facts.

5.3.1 Preliminary statements.

Our proof is mainly based on "Kotani's formula", expressed in [53],

Statement 5.3.1 (Kotani's lemma) *Let $\lambda > 0$. Then for $t \geq 0$*

$$E_W\left[e^{-\lambda H(t)}\right] = \exp\left(-2\lambda \int_0^t U_\lambda(s) ds\right), \ P - a.s.,$$

where $U_\lambda(t)$ is the unique stationnary and positive solution of the equation

$$dU_\lambda(t) = U_\lambda(t)dW(t) + \left(1 + \frac{1-\kappa}{2}U_\lambda(t) - 2\lambda U_\lambda(t)^2\right)dt.$$

(Here $W(t)$ is the Brownian motion defined in the introduction).

We shall also use the following result from [41] (Lemma 2.4)

Statement 5.3.2

$$\lim \frac{1}{r} \sup_{|x|<u} \left(L^x_{\tau(r)} - r\right) = 0,\ a.s.,$$

whenever $u \to \infty$ and $r \gg u \log\log u$.

5.3.2 Proof of Theorem 2.6.4.

We use the same time change method as in the annealed case, in order to get almost sure estimates for U_λ. Let

$$g(x) = \int_1^x \frac{e^{2/s+4\lambda s}}{s^{1-\kappa}} ds.$$

One can easily check that g is a scale function of U_λ. By the same arguments as in section 5.1.2, we get

$$\int_0^t U_\lambda(s)ds = \int_0^{\mu(t)} g^{-1}(\gamma(u))^{1-2\kappa} \exp\left(-\frac{4}{g^{-1}(\gamma(u))} - 8\lambda g^{-1}(\gamma(u))\right)du,$$

where $\gamma(u)$ is a standard brownian motion,

$$\mu(t) = \int_0^t U_\lambda(s)^{2\kappa} \exp\left(\frac{4}{U_\lambda(s)} + 8\lambda U_\lambda(s)\right)ds,$$

and

$$\mu^{-1}(t) = \int_0^t g^{-1}(\gamma(s))^{-2\kappa} \exp\left(-\frac{4}{g^{-1}(\gamma(s))} - 8\lambda g^{-1}(\gamma(s))\right)ds.$$

We have the following lemma, whose proof is postponed

Lemma 5.3.1 *Let $\nu \in \mathbb{R}$, and*

$$D_\nu(r) = \int_0^{\tau_r} g^{-1}(\gamma(s))^\nu \exp\left(-\frac{4}{g^{-1}(\gamma(s))} - 8\lambda g^{-1}(\gamma((s))\right)ds.$$

Then, whenever $\lambda \to 0$ and $r \gg \log(1/\lambda)\log\log(1/\lambda)$,

$$D_{1-2\kappa}(r) = r(1+o(1))\frac{\Gamma(1-\kappa)}{(4\lambda)^{1-\kappa}};$$

and for some positive constant D,

$$D_{-2\kappa}(r) = Dr(1 + o(1)).$$

Let us use this lemma to finish the proof of Theorem 2.6.4. We get easily that $\mu^{-1}(\tau_r) = D_{-2\kappa}(r)$. Whence, for some constant D',

$$\tau_{D'(1-o(1))t} \leq \mu(t) \leq \tau_{D'(1+o(1))t}$$

almost surely, as $\lambda \to 0$ and $t \gg \log(1/\lambda)\log\log(1/\lambda)$. Therefore, under the same assumptions, for some constant D'',

$$D''(1 - o(1))t\frac{\Gamma(1-\kappa)}{(4\lambda)^{1-\kappa}} \leq \int_0^t U_\lambda(s)ds \leq D''(1 + o(1))t\frac{\Gamma(1-\kappa)}{(4\lambda)^{1-\kappa}}.$$

Thus, going back to Kotani's lemma, for $t > 0$, and for some constant C, we get, as $\lambda \to 0$, $v \gg \log(1/\lambda)\log\log\log(1/\lambda)$,

$$\exp\left(-C(1 + o(1))\lambda^\kappa v\right) \leq E_W\left[e^{-\lambda H(v)}\right] \leq \exp\left(-C(1 - o(1))\lambda^\kappa v\right),\ P - a.s.. \tag{5.3.1}$$

By application of Chebyshev's inequality, for λ as before,

$$\log P_W\left[H(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right] \leq \lambda\left(\frac{v}{u}\right)^{1/\kappa} - C(1 - o(1))v\lambda^\kappa.$$

We call $\lambda(x)$ the value of lambda that minimizes $\lambda x - Cv\lambda^\kappa$. It is clear that $\lambda(x)$ is a decreasing function of x, such that

$$\lambda(x)x = Cv\kappa\lambda(x)^\kappa. \tag{5.3.2}$$

Let $\lambda^* = \lambda\left(\left(\frac{v}{u}\right)^{1/\kappa}\right)$, we get easily the expression

$$\lambda^{*\kappa} = (C\kappa)^{\frac{\kappa}{1-\kappa}}\frac{u^{\frac{1}{1-\kappa}}}{v}. \tag{5.3.3}$$

One can easily check that $\lambda^* \to 0$, $v \gg \log(1/\lambda^*)\log\log\log(1/\lambda^*)$. Therefore we can apply the precedent estimate to get

$$\limsup_{v\to\infty}\frac{\log P_W\left[H(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right]}{u^{\frac{1}{1-\kappa}}} \leq (\kappa - 1)C^{\frac{1}{1-\kappa}}\kappa^{\frac{\kappa}{1-\kappa}}.$$

In order to get the lower bound, we introduce a small $\delta > 0$. For the sake of clarity we call $\varepsilon := \left(\frac{v}{u}\right)^{1/\kappa}$. Note that for $\lambda > 0$

$$
\begin{aligned}
E_W\left[e^{-\lambda^* H(v)}\right] = E_W\left[e^{-\lambda^* H(v)}\mathbf{1}_{H(v)<(1-\delta)\varepsilon}\right] \\
+ E_W\left[e^{-\lambda^* H(v)}\mathbf{1}_{(1-\delta)\varepsilon \le H(v) \le (1+\delta)\varepsilon}\right] + E_W\left[e^{-\lambda^* H(v)}\mathbf{1}_{H(v)>(1+\delta)\varepsilon}\right] \\
:= J_1 + J_2 + J_3.
\end{aligned}
$$

We are going to show that $J_1 + J_3 \ll E_W\left[e^{-\lambda^* H(v)}\right]$. We call $F(x) = P_W[H(v) < x]$. By the Cramer-Chernoff inequality, for $x < \varepsilon$, one gets

$$
\begin{aligned}
F(x) &\le \exp\left(\lambda(x)x - C(1-o(1))v\lambda(x)^\kappa\right) \\
&= \exp\left(-C(1-o(1))v(1-\kappa)\lambda(x)^\kappa\right) \\
&= \exp\left[-(1-o(1))C^{\frac{1}{1-\kappa}}(1-\kappa)\kappa^{\frac{\kappa}{1-\kappa}}v^{\frac{1}{1-\kappa}}x^{\frac{\kappa}{\kappa-1}}\right]. \quad (5.3.4)
\end{aligned}
$$

Recall that

$$E\left[e^{-\lambda^* H(v)}\right] = e^{-C(1+o(1))(C\kappa)^{\frac{\kappa}{1-\kappa}}u^{\frac{1}{1-\kappa}}}.$$

We deduce that for $\alpha = 2(1-\kappa)^{\frac{\kappa-1}{\kappa}}$,

$$F(\alpha\varepsilon) \ll E_W\left[e^{-\lambda^* H(v)}\right].$$

For this α, we have

$$
\begin{aligned}
J_1 &\le F(\alpha\varepsilon) + \int_{\alpha\varepsilon}^{(1-\delta)\varepsilon} e^{-\lambda^* x} dF(x) \\
&= e^{-(1-\delta)\varepsilon\lambda^*}F((1-\delta)\varepsilon) + (1-e^{-\alpha\varepsilon})F(\alpha\varepsilon) + \lambda^*\int_{\alpha\varepsilon}^{(1-\delta)\varepsilon} e^{-\lambda^* x}F(x)dx. \quad (5.3.5)
\end{aligned}
$$

Our goal is to use (5.3.4) in order to bound F in the last equation. The problem is that the $o(1)$ in (5.3.4) depends on x. We are going to use the monotonicity of $F(x)$ in order to get an uniform bound. Let $\eta < \delta/1000$, $n > \frac{\kappa}{\alpha(1-\kappa)\eta}$. For $1 \le k \le n$, we set $x_k = k\varepsilon/n$. Using (5.3.4), there exists v_0 such that, for all $v > v_0$, and $1 \le k \le n$,

$$F(x_k) \le \exp\left[-(1-\eta)C^{\frac{1}{1-\kappa}}(1-\kappa)\kappa^{\frac{\kappa}{1-\kappa}}v\left(\frac{x_k}{v}\right)^{\frac{\kappa}{\kappa-1}}\right].$$

Note that for $x_{k-1} < x < x_k$, $x > \alpha\varepsilon$, and $v > v_0$,

$$F(x) \leq F(x_\kappa) \leq \exp\left[-(1-\eta)C^{\frac{1}{1-\kappa}}(1-\kappa)\kappa^{\frac{\kappa}{1-\kappa}}v^{\frac{1}{1-\kappa}}\left(x+\frac{\varepsilon}{n}\right)^{\frac{\kappa}{\kappa-1}}\right].$$

By the concavity of the function $x \to x^{\frac{\kappa}{\kappa-1}}$, and the condition $\varepsilon > x > \alpha\varepsilon$, we get easily

$$\left(x+\frac{\varepsilon}{n}\right)^{\frac{\kappa}{\kappa-1}} \geq x^{\frac{\kappa}{\kappa-1}} + \frac{1}{n}\frac{\kappa}{\alpha(\kappa-1)}\alpha^{\frac{\kappa}{\kappa-1}}\varepsilon^{\frac{\kappa}{\kappa-1}} \geq (1-\eta)x^{\frac{\kappa}{\kappa-1}}.$$

We deduce that for every $\varepsilon > x > \alpha\varepsilon$,

$$F(x) \leq \exp\left[-(1-\eta)^2C^{\frac{1}{1-\kappa}}(1-\kappa)\kappa^{\frac{\kappa}{1-\kappa}}v^{\frac{1}{1-\kappa}}x^{\frac{\kappa}{\kappa-1}}\right] := e^{G(x)} \tag{5.3.6}$$

Therefore, replacing F by e^G in (5.3.5), and doing the integration by parts in the other direction, we get

$$\begin{aligned} J_1 &\leq e^{-(1-\delta)\varepsilon\lambda^*}e^{G((1-\delta)\varepsilon)} + (1-e^{-\alpha\varepsilon})e^{G(\alpha\varepsilon)} + \lambda^*\int_{\alpha\varepsilon}^{(1-\delta)\varepsilon} e^{-\lambda^* x}e^{G(x)}dx \\ &= e^{G(\alpha\varepsilon)} + \int_{\alpha\varepsilon}^{(1-\delta)\varepsilon} e^{-\lambda^* x}de^{G(x)}. \end{aligned} \tag{5.3.7}$$

Recalling the definition of α,

$$e^{G(\alpha\varepsilon)} \ll E\left[e^{-\lambda^* H(v)}\right],$$

and the integral can be bounded by

$$C'v^{\frac{\kappa}{1-\kappa}}\int_{\alpha\varepsilon}^{(1-\delta)\varepsilon} x^{\frac{1}{\kappa-1}}e^{-\lambda^* x}e^{G(x)}dx.$$

Therefore, recalling (5.3.1), and (5.3.3) for estimates on $E_W\left[e^{-\lambda^* H(v)}\right]$, and the expressions of $\lambda(x)$ and G respectively in (5.3.2) and (5.3.6), one gets

$$\begin{aligned} J_1\left(E_W\left[e^{-\lambda^* H(v)}\right]\right)^{-1} \leq o(1) + P\sup_{x\in[\alpha\epsilon,(1-\delta)\epsilon]}\exp\Big(&-C^{\frac{1}{1-\kappa}}\kappa^{\frac{\kappa}{1-\kappa}}v^{\frac{1}{1-\kappa}} \\ &\left[(1-\eta)^2(1-\kappa)x^{\frac{\kappa}{1-\kappa}} + \kappa\varepsilon^{\frac{1}{\kappa-1}} - \varepsilon^{\frac{\kappa}{\kappa-1}}(1+o(1))\right]\Big). \end{aligned}$$

where P is some polynome in (u, v) and the terms between the brackets come respectively from e^G, $e^{-\lambda x}$ and $\left(E_W\left[e^{-\lambda^* H(v)}\right]\right)^{-1}$. By a change of variable in the sup, we get

$$\begin{aligned} \frac{J_1}{E_W\left[e^{-\lambda^* H(v)}\right]} &< o(1)+ \\ &P\exp\left(-(Cv)^{\frac{1}{1-\kappa}}(\varepsilon\kappa)^{\frac{\kappa}{1-\kappa}}\inf_{s\in[\alpha,(1-\delta)]}\left[(1-\eta)^2 s^{\frac{\kappa}{\kappa-1}}(1-\kappa) + \kappa s - 1 + o(1)\right]\right). \end{aligned} \tag{5.3.8}$$

For η and $o(1)$ very small,

$$\inf_{s\in[\alpha,(1-\delta)]}\left[(1-\eta)^2 s^{\frac{\kappa}{\kappa-1}}(1-\kappa)+\kappa s-1+o(1)\right]$$

is positive by concavity of the function $s\to s^{\frac{\kappa}{\kappa-1}}(1-\kappa)$, therefore as an easy consequence

$$J_1 \ll E_W\left[e^{-\lambda^* H(v)}\right].$$

We now deal with J_3. As before we get

$$J_3 < e^{-(1+\delta)\varepsilon\lambda^*}F((1+\delta)\varepsilon)+\lambda^*\int_{(1+\delta)\varepsilon}^{\infty} e^{-\lambda^* x}F(x)dx$$

for $\beta>0$, as $F(x)\le 1$

$$\lambda^*\int_{\beta\varepsilon}^{\infty} e^{-\lambda^* x}F(x)dx \le e^{-\beta\lambda^*\varepsilon}=\exp\left(-\beta(C\kappa)^{\frac{1}{1-\kappa}}u^{\frac{1}{1-\kappa}}\right)$$

therefore for some β depending on κ,

$$R(\varepsilon):=\lambda^*\int_{\beta\varepsilon}^{\infty} e^{-\lambda^* x}F(x)dx \ll E_W\left[e^{-\lambda^* H(v)}\right].$$

by the same argument as for J_1, we get that, for any $\varepsilon<x<\beta\varepsilon$, for v large enough,

$$F(x)\le e^{G(x)};$$

therefore

$$J_3-R(\varepsilon)\le e^{-(1+\delta)\varepsilon\lambda^*}e^{G((1+\delta)\varepsilon)}+\lambda^*\int_{(1+\delta)\varepsilon}^{\beta\varepsilon} e^{-\lambda^* x}e^{G}(x)dx$$

By the same computation as we did to get to (5.3.8), we have

$$\begin{aligned}\frac{J_3}{E_W\left[e^{-\lambda^* H(v)}\right]} &< o(1)+\\ &P\exp\left(-(Cv)^{\frac{1}{1-\kappa}}(\varepsilon\kappa)^{\frac{\kappa}{1-\kappa}}\inf_{s\in[(1+\delta,\beta]}\left[(1-\eta)^2 s^{\frac{\kappa}{\kappa-1}}(1-\kappa)+\kappa s-1+o(1)\right]\right). \quad (5.3.9)\end{aligned}$$

As before, we can take η small and get

$$J_3 \ll E_W\left[e^{-\lambda^* H(v)}\right].$$

Therefore we get that, as $v\to\infty$,

$$J_2 > \frac{1}{2}E_W\left[e^{-\lambda^* H(v)}\right].$$

Recall that

$$J_2 = E_W\left[e^{-\lambda^* H(v)}\mathbf{1}_{(1-\delta)\varepsilon\leq H(v)\leq(1+\delta)\varepsilon}\right] \leq e^{-\lambda*(1-\delta)\varepsilon}P_W\left[H(v) < (1+\delta)\varepsilon\right].$$

Note that the preceding computations remain true for $u' := (1+\delta)^\kappa u$, whence

$$\liminf_{v\to\infty}\frac{\log P_W\left[H(v) < \left(\frac{v}{u}\right)^{1/\kappa}\right]}{u^{\frac{1}{1-\kappa}}} > (1+\delta)^{\frac{\kappa}{1-\kappa}}((1-\delta)\kappa-1)C^{\frac{1}{1-\kappa}}\kappa^{\frac{\kappa}{1-\kappa}}.$$

Taking the limit as $\delta\to 0$, we get the result.

It remains to prove lemma 5.3.1, which is the purpose of the next section.

5.3.3 Proof of Lemma 5.3.1.

Let $\nu = 1-2\kappa$, and

$$\begin{aligned} D_\nu = \int_0^{\tau_r} g^{-1}(\gamma(s))^\nu \exp\left(-\frac{4}{g^{-1}(\gamma(s))} - 8\lambda g^{-1}(\gamma(s))\right)ds. \\ = \int_{-\infty}^{\infty} g^{-1}(s)^\nu \exp\left(-\frac{4}{g^{-1}(s)} - 8\lambda g^{-1}(s)\right)L^s_{\tau_r}ds. \\ = \left(\int_{-\infty}^0 + \int_0^{g(a)} + \int_{g(a)}^{\infty}\right) g^{-1}(s)^\nu \exp\left(-\frac{4}{g^{-1}(s)} - 8\lambda g^{-1}(s)\right)L^s_{\tau_r}ds \\ := I_1 + I_2 + I_3, \end{aligned}$$

where a is such that $a > 1/\lambda$ and

$$\frac{e^{4\lambda a}}{4\lambda a} = \log\frac{1}{\lambda}\log\log\log\frac{1}{\lambda}.$$

We shall use the following consequence of the law of large numbers : let $f:\mathbb{R}\to\mathbb{R}$ such that $\int_{\mathbb{R}}|f(x)|dx < \infty$, then

$$\lim_{r\to\infty}\frac{1}{r}\int_{\mathbb{R}} f(x)L^x_{\tau_r}dx = \int_{\mathbb{R}} f(x)dx. \tag{5.3.10}$$

Note that, for $x < 1$ and $\lambda < 1/4$,

$$|g(x)| = \int_x^1 \frac{e^{2/s+4\lambda s}}{s^{1-\kappa}}ds \leq \frac{e^{2/x+1}}{x^{1-\kappa}},$$

therefore, for some constant $c > 0$, for all $x\leq 0$ and $\lambda < 1/4$ we have

$$\frac{2}{g^{-1}(x)} \geq \log\frac{|x|}{c}. \tag{5.3.11}$$

On the other hand

$$I_1 = \int_{-\infty}^{0} g^{-1}(s)^{\nu} \exp\left(-\frac{4}{g^{-1}(s)} - 8\lambda g^{-1}(s)\right) L^s_{\tau_r} ds$$
$$\leq \int_{-\infty}^{0} g^{-1}(s)^{\nu} \exp\left(-\frac{4}{g^{-1}(s)}\right) L^s_{\tau_r} ds.$$

Using (5.3.11), it is not difficult to check that $g^{-1}(s)^{\nu} \exp\left(-\frac{4}{g^{-1}(s)}\right)$ is integrable on $(-\infty, 0)$, therefore an application of (5.3.10) lays

$$I_1 = O(r).$$

Let us now treat I_3. Note that for $y \geq a$, $y\lambda \to \infty$ and for some constant $c > 0$

$$\frac{1}{c}\int_1^y \frac{e^{4\lambda s}}{s^{1-\kappa}} ds \leq g(y) \leq c\int_1^y \frac{e^{4\lambda s}}{s^{1-\kappa}} ds$$

and

$$\int_1^y \frac{e^{4\lambda s}}{s^{1-\kappa}} ds = \frac{1}{(4\lambda)^{\kappa}} \int_{4\lambda}^{4\lambda y} \frac{e^s}{s^{1-\kappa}} ds = (1+o(1))\left(\frac{e^{4\lambda y}}{4\lambda y}\right). \tag{5.3.12}$$

As $y\lambda \to \infty$, we get

$$g(y) \leq 2ce^{4\lambda y}.$$

Therefore, for $x \geq g(a)$, $2ce^{4\lambda g^{-1}(x)} \geq x$, so $g^{-1}(x) \geq \frac{1}{4\lambda}\log\frac{x}{2c}$. Therefore, using (5.3.10) we get, for some constant $c' > 0$,

$$I_3 \leq \int_{g(a)}^{\infty} (g^{-1}(x))^{\nu\vee 0} e^{-8\lambda g^{-1}(x)} L^x_{\tau_r} dx \leq c' \int_{g(a)}^{\infty} \left(\frac{\log(x/2c)}{4\lambda}\right)^{\nu\vee 0} x^{-2} L^x_{\tau_r} dx$$
$$\leq c'(g(a))^{-1/2} \int_1^{\infty} \left(\frac{\log(x/2c)}{4\lambda}\right)^{\nu\vee 0} x^{-3/2} L^x_{\tau_r} dx = o\left(\frac{r}{\lambda^{\nu\vee 0}}\right).$$

To deal with I_2, note that, by the definition of a and (5.3.12),

$$r \gg g(a)\log\log g(a).$$

Therefore we can apply statement 5.3.2 to get

$$I_2 = r(1+o(1)) \int_0^{g(a)} g^{-1}(s)^{\nu} \exp\left(-\frac{4}{g^{-1}(s)} - 8\lambda g^{-1}(s)\right) ds.$$

By a change of variables $g^{-1}(s) = y$, as $\lambda \to 0$, the last integral is equal to

$$\int_1^a \frac{e^{-\frac{2}{y}-4\lambda y}}{y^{1-\kappa-\nu}} dy = (1+o(1))\frac{1}{(4\lambda)^{\kappa+\nu}} \int_{4\lambda}^{4\lambda a} \frac{e^{-u}}{u^{1-(\nu+\kappa)}} du.$$

Recalling the definition of ν we have $\nu + \kappa = 1 - \kappa > 0$, then

$$I_2 = r(1 + o(1)) \frac{\Gamma(1-\kappa)}{(4\lambda)^{1-\kappa}}.$$

This finishes the proof of the first part of lemma 5.3.1, as $1 - \kappa > \nu \vee 0$.
To treat the case $\nu = -2\kappa$, let $b < 1$ be such that $b \to 0$ and $-g(b) = o\left(\frac{r}{\log\log r}\right)$. As before, we separate the integral as follows

$$D_\nu = \left(\int_{-\infty}^{g(b)} + \int_{g(b)}^{g(a)} + \int_{g(a)}^{\infty} \right) g^{-1}(s)^\nu \exp\left(-\frac{4}{g^{-1}(s)} - 8\lambda g^{-1}(s) \right) L^s_{\tau_r} ds$$
$$:= I'_1 + I'_2 + I'_3.$$

I'_3 is similar to the precedent case, with $\nu < 0$, so we get $I'_3 = o(r)$. We have easily

$$I'_1 \leq e^{-\frac{1}{b}} \int_{-\infty}^{0} g^{-1}(s)^\nu \exp\left(-\frac{3}{g^{-1}(s)} - 8\lambda g^{-1}(s) \right) L^s_{\tau_r} ds.$$

The integral is a $O(r)$ by the same proof as for I_1, therefore

$$I'_1 = o(r).$$

By the same proof as for I_2, we get

$$I'_2 = r(1 + o(1)) I''_2,$$

with

$$I''_2 = \int_b^a \frac{e^{-\frac{2}{y} - 4\lambda y}}{y^{1+\kappa}} dy = \int_b^1 \frac{e^{-\frac{2}{y} - 4\lambda y}}{y^{1+\kappa}} dy + \int_1^a \frac{e^{-\frac{2}{y} - 4\lambda y}}{y^{1+\kappa}} dy.$$

The first part converges, by dominated convergence, to

$$D := \int_0^1 \frac{e^{-\frac{2}{y}}}{y^{1+\kappa}} dy,$$

and the second part is equal to

$$(4\lambda)^\kappa \int_{4\lambda}^{4\lambda a} \frac{e^{-8\lambda/u - u}}{u^{1+\kappa}} du.$$

One can easily check that the integral is bounded, therefore this part goes to zero. This finishes the proof of lemma 5.3.1.

5.3.4 Quenched Speedup for the diffusion.

In this section we prove Theorem 2.6.3. The upper bound is a trivial consequence of Theorem 2.6.4 , since

$$P_W[X_t > t^\kappa u] \leq P_W[H(t^\kappa u) < t].$$

To get the lower bound, let $\varepsilon > 0$. Note that

$$P_W[X_t > t^\kappa u] \geq P_W[H((1+\varepsilon)t^\kappa u) < t] P_W^{(1+\varepsilon)t^\kappa u}[H(t^\kappa u) > t].$$

Note that almost surely, for t large enough, we can find $t^\kappa u < b < c < (1+\varepsilon)t^\kappa u$ such that

$$W_\kappa(b) - W_\kappa(c) > \frac{\varepsilon\kappa}{2} t^\kappa u.$$

It is clear that

$$P_W^{2t^\kappa u}[H(t^\kappa u) > t] \geq P_W^c[H(b) > t].$$

By the same computations as in 5.2.2, one gets easily that

$$P_W^c[H(b) > t] > 1/2$$

for t large enough. Taking the limit as $\varepsilon \to 0$, this finishes the proof of Theorem 2.6.3.

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Sur certains processus aléatoires en milieu aléatoire

Résumé : Cette thèse a pour objet l'étude de processus aléatoires en milieu aléatoire. Ce type de processus a été introduit pour la première fois en 1965 par A.A. Chernov, et a depuis fait l'objet de nombreuses recherches. Parallèlement au modèle élémentaire unidimensionnel étudié par A.A. Chernov, de nombreuses tentatives ont été faites récemment afin d'appliquer le même type d'approche dans des contextes différents. Nous nous focalisons particulièrement sur deux exemples. Tout d'abord nous étudions le cas de la marche aléatoire en milieu aléatoire sur les arbres, pour laquelle nous étendons un critère de récurrence/transience dû à R. Lyons et R. Pemantle, avant de présenter une étude du comportement asymptotique dans le régime critique. Nous montrons dans un premier cas un théorème central limite, et dans un second nous identifions un équivalent en $(\log n)^3$. Le régime intermédiaire entre ces deux comportements a fait l'objet de travaux plus anciens de Y. Hu et Z. Shi.

Dans une autre partie nous étudions un processus aléatoire en milieu aléatoire à temps continu, connu sous le nom de diffusion de Brox. Nous étendons à ce processus des résultats dûs à A. Fribergh, N. Gantert et S. Popov concernant l'accélération et le ralentissement.

On some processes in a random environment

Abstract : This thesis's aim is to study random processes in a random environment. This kind of processes was introduced for the first time in 1965 by A.A. Chernov, and has received much attention since. Besides the elementary unidimensional model studied by A.A. Chernov, numerous attempts have been made in order to apply a similar approach to different contexts. We focus in particular on two examples. First we study the case of random walk in a random environment on trees, for which we extend a recurrence/transience criterion due to R. Lyons and R. Pemantle, before studying the asymptotic behavior in the critical case. In a first case we give a central limit theorem, while in a second one we show that the walk behave like $(\log n)^3$. In the intermediate regime between those two behaviors, we refer to previous work by Y. Hu and Z. Shi.

In a second part we study a time continuous process in a random environment, known as Brox's diffusion. We extend to this context some recent results due to A. Fribergh, N. Gantert and S. Popov about speedup and slowdown.

Discipline : Mathématiques

Mots clés: Random processes, Random environment, Markov chains, Brownian Motion, Central Limit Theorem, Trees, Branching processes, Diffusions

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